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AUTHOR Anderson, R. D.; And Others
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ABSTRACT

This eighth unit in the SMSG junior high mathematics series is the teacher's commentary for Unit 6. A time allotment for each of the chapters in Unit 6 is suggested. Then, for each of the chapters in Unit 6, the objectives for that chapter are specified, the mathematics is discussed, some teaching suggestions are provided, the answers to exercises are listed, and sample test questions for that chapter are suggested. (DT)

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TEACHER'S COMMENTARY

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PART II

SMSG

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2

School Mathematics Study Group

Mathematics for Junior High School, Volume 2

Unit 8

Mathematics for Junior High School, Volume 2

Teacher's Commentary, Part II

Prepared under the supervision of the
Panel on Seventh and Eighth Grades
of the School Mathematics Study Group:

R. D. Anderson	Louisiana State University
J. A. Brown	University of Delaware
Lenore John	University of Chicago
B. W. Jones	University of Colorado
P. S. Jones	University of Michigan
J. R. Mayor	American Association for the Advancement of Science
P. C. Rosenbloom	University of Minnesota
Veryl Schult	Supervisor of Mathematics, Washington, D.C.

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The following is a list of all those who have participated in the preparation of this volume.

R.D. Anderson, Louisiana State University
B.H. Arnold, Oregon State College
J.A. Brown, University of Delaware
Kenneth E. Brown, U.S. Office of Education
Mildred B. Cole, K.D. Waldo Junior High School, Aurora, Illinois
B.H. Colvin, Boeing Scientific Research Laboratories
J.A. Cooley, University of Tennessee
Richard Dean, California Institute of Technology
H.M. Gehman, University of Buffalo
L. Roland Genise, Brentwood Junior High School, Brentwood, New York
E. Glenadine Gibb, Iowa State Teachers College
Richard Good, University of Maryland
Alice Hach, Ann Arbor Public Schools, Ann Arbor, Michigan
S.B. Jackson, University of Maryland
Lenore John, University High School, University of Chicago
B.W. Jones, University of Colorado
P.S. Jones, University of Michigan
Houston Karnes, Louisiana State University
Mildred Keiffer, Cincinnati Public Schools, Cincinnati, Ohio
Nick Lovdjieff, Anthony Junior High School, Minneapolis, Minnesota
J.R. Mayor, American Association for the Advancement of Science
Sheldon Meyers, Educational Testing Service
Muriel Mills, Hill Junior High School, Denver, Colorado
P.C. Rosenbloom, University of Minnesota
Elizabeth Roudebush Mitchell, Seattle Public Schools, Seattle, Washington
Veryl Schult, Washington Public Schools, Washington, D.C.
George Schaefer, Alexis I. DuPont High School, Wilmington, Delaware
Allen Shields, University of Michigan
Rothwell Stephens, Knox College
John Wagner, School Mathematics Study Group, New Haven, Connecticut
Ray Walch, Westport Public Schools, Westport, Connecticut
G.C. Webber, University of Delaware
A.B. Willcox, Amherst College

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NOTE TO TEACHERS

Based on the teaching experience of over 100 junior high school teachers in all parts of the country and the estimates of authors of the revisions (including junior high school teachers), it is recommended that teaching time for Part 2 be as follows:

Chapter		Approximate number of days
7		13
8		9-10
9		15
10		10
11		13
12		10-11
13		7
	Total	77-79

Teachers are urged to try not to exceed these approximate time allotments so that pupils will not miss the chapters at the end of the course. Some classes will be able to finish certain chapters in less than the estimated time.

Throughout the text, problems, topics, and sections which were designed for the better students are indicated by an asterisk (*). Items starred in this manner should be used or omitted as a means of adjusting the approximate time schedule.

Several supplementary units are available for self-study by exceptional students or for class use in advanced sections.

Chapter 7

PERMUTATIONS AND SELECTIONS

General Remarks

The study of permutations and selections (or combinations) is a "fresh start" for many students because nearly all that is required as a background is a working knowledge of the four operations with numbers and an understanding of formulas. It might be well to call this fact to the attention of certain pupils who may be able but not always willing.

Committee arrangements are used in the beginning because the familiarity of pupils with this type of situation ensures that the only new ideas introduced are those of a purely mathematical nature. Other varied and interesting applications of the mathematical principles learned in this chapter are found in the exercises. The authors prefer to use the word "selection" rather than "combination" (although both words are found in the text). It is believed that "selection" would be a more natural and meaningful term for those working with these ideas for the first time.

Pascal's triangle, as introduced in Section 1, not only is an exciting application of selections, but also serves other purposes. It is an excellent device for helping students recall quickly the number of selections that can be made from n things r at a time. Having this information about selections readily available speeds up class discussion and allows for development of concepts without interrupting the train of thought with constant, and sometimes irrelevant, calculations. Also the patterns involved in the triangle suggest certain inductive conclusions about the number relations among the various entries.

Contrary to what is probably the usual procedure, combinations (selections) are used as an introduction to permutations. By ordering the members of a subcommittee and applying the principles learned in the study of selections, students are prepared to accept and understand the following property: If n is a counting number, then the number of permutations of any set of n numbers is $n!$. The study of permutations in this chapter does not include those groups in which only a part of the items are repeated, such as the permutations of the letters of the word, "Mississippi."

Some sections contain a list of Class Exercises which are planned for developmental class discussions and are not assignment exercises. The intention is not to limit discussion to the specific questions listed. It is expected that the teacher will use this list of questions merely as a guide. However, each question is carefully placed for developmental purposes in the order given and is pointed toward a particular idea in the developmental sequence. If this material is relatively unfamiliar to the teacher, it is recommended that the sequence be adhered to as given. The exercises at the end of each part of a section are thought to be sufficient for one assignment.

It is hoped that the study of this chapter will widen pupils' appreciation for mathematics as a science. The investigation of Pascal's triangle begins with a nearly complete body of facts, includes the discovery and observation of quantitative relations, inductively suggests certain principles, verifies these deductively, and derives generalizations. It is thought that the choice of Pascal's triangle might be a good one because the body of facts is small, the quantitative relations are of a very simple nature, the discovery of the patterns involved is not difficult, and the number of experiments that may be performed is ample.

It is also hoped that the study of this chapter will prove to be a source of intellectual satisfaction and fun for the pupil and will help to strengthen his realization of the power of mathematics. He may be impressed by the vast amount of hard work that would be required to determine the third term in the two-thousandth line of Pascal's triangle if he were actually to write down all the numbers in all the rows. He should be equally impressed by the ease with which the same information may be obtained by using $\frac{n(n-1)}{2}$.

For better students, reference may be made to Section 1 and to Problem 6, Exercise 3b of the supplementary unit on "Finite Differences."

A time allowance of 13 class periods is suggested for this chapter.

7-1. The Pascal Triangle

In this section, as in those to follow, the ideas are developed through carefully selected class discussion exercises. The questions in these exercises should be answered by students as they manipulate objects or write symbols for the elements of a set. Tabulating and counting are important key notions to be used throughout the section. Many students will prefer to use semi-concrete representations provided by writing symbols on paper but some use might also be made of concrete objects to be grouped and counted.

Perhaps the title for this section will not be very useful until the study of the section has been completed. If questions are asked about the title, no harm will be done in displaying a triangle of three or four rows during the first discussion session. Some of the interesting properties of Pascal's triangle might even be pointed out at that time.

One of the important ideas to be brought out in this study is that if 3 objects have been selected from 5 then a selection of 2 objects has also been made. The symmetry of Pascal's triangle is based on this property. The student should clearly understand that the number of ways of selecting 17 objects from a set of 26 objects is exactly the same as the number of ways of selecting 9 objects from a set of 26 objects.

[pages 281-288]

Students should be encouraged to observe that each entry in Pascal's triangle is the sum of the two numbers nearest to it on the preceding line. The better students, by the end of the chapter at least, may be able to state why this property is true. One way of showing this property follows. Consider 5 lines of Pascal's triangle:

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Why is the number of combinations of 5 things taken three at a time the sum of the number of combinations of 4 things taken 2 at a time and the number of combinations of 4 things taken 3 at a time? Think of 5 letters A, B, C, D, E. If we select 3 of the 4 letters B, C, D, E, then of course, we have part of the sets of 3 of the 5 letters A, B, C, D, E. The number we have is $\binom{4}{3}$. The ones we have not counted are the groups of 3 of which A is one letter of the group. Any group of 3 including A consists of A and 2 of the 4 letters B, C, D, E. The number of groups of 2 from four letters is $\binom{4}{2}$. Hence,

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}.$$

Or in more general terms, let us try to show

$$\binom{n}{3} = \binom{n-1}{3} + \binom{n-1}{2}.$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \qquad \binom{n-1}{3} = \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3}$$

$$\binom{n-1}{2} = \frac{(n-1)(n-2)}{1 \cdot 2}$$

[pages 281-288]

$$\binom{n-1}{3} + \binom{n-1}{2} = \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} + \frac{(n-1)(n-2)}{1 \cdot 2} \cdot \frac{3}{3}$$

$$= \frac{(n-1)(n-2)(n-3+3)}{1 \cdot 2 \cdot 3}$$

$$= \frac{(n-1)(n-2)n}{1 \cdot 2 \cdot 3}$$

But $\binom{n}{3} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$.

We have proved only a special case. Other special cases could be proved similarly. In order to prove the general property,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

we would have to use mathematical induction.

The better students may also be interested in other properties of Pascal's triangle. The sum of the numbers in any row is always a power of 2; not considering the 1's, the greatest common factor in any prime-numbered row is the number of the row; the third term in each row can be predicted by using the formula, $\frac{n(n-1)}{2}$, and the fourth term in each row can be predicted by using the formula, $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$. Reference will be made to Pascal's triangle in the sections to follow and even though no student observes these properties now, several may do so as the work progresses.

Particularly in this first section the treatment is intended to be quite informal. It would be better that no formulas be introduced into the general class discussions. However, the statement of generalizations which can be illustrated in special cases should be encouraged.

Answers to Class Exercises 7-1a

1. {A,D,E}
2. Yes
3. {B,A,E}; {E,B,A}
4. {ABC}, {ABD}, {ABE}, {ACD}, {ACE}, {BCD}, {BCE}. {CDE}, {DEA}, {DEB}
5. 10
6. 6
7. B and D are on the same number of committees.
8. 4
9. $\frac{6}{10}$ or $\frac{3}{5}$
10. 3
11. $\frac{3}{6}$ or $\frac{1}{2}$
12. {B,C,D}
13. A and D
14. 10
15. 6
16. 4
17. {C,A}; {C,B}; {C,D}; {C,E}

Answers to Exercises 7-1a

1. (a) {A,B,C,D}, {A,B,C,E}, {A,B,D,E}, {A,C,D,E}, {B,C,D,E}
   (b) 1
   (c) {A}, {B}, {C}, {D}, {E}
   (d) Same.
   (e) 1
2. (a) 1
   (b) 4
   (c) {K}, {L}, {M}, {N}

[pages 281-283]

(d) The three-member committee is composed of all the club members who are not in the corresponding one-member committee.

(e) {L,M,N}, {K,M,N}, {K,L,N}, {K,L,M}

(f) {K,L}, {K,M}, {K,N}, {L,M}, {L,N}, {M,N}

3. (a) There is one committee with three members, namely, {P,Q,R}; there are three committees with two members, namely, {Q,R}, {P,R}, {P,Q}; there are three committees with one member, namely, {P}, {Q}, {R}.

4. The committees are {U,V}, {U}, {V}.

5. If the club member is Z, the only committee is {Z}.

6. 6

7. 10

Answers to Class Exercises 7-1b

1. One committee with four members.

2. One committee with no members at all.

3. 1, 3, 3, 1.

4. One committee with no members.
Two committees with one member.
One committee with two members.

5. 1, 1. The second 10 tells how many committees with three members each.

6. 6

7. Two vacancies. Four.

8. Also 6.

9. 4

10. Three vacancies. 4.

11. 3, 4.

Answers to Exercises 7-1b

2. (a) The number of possible committees with 2 members formed from a club of 4 members.

   (b) The number of possible committees with 1 member formed from a club of 3 members.

3. (a) The number of possible committees with 4 members formed from a club of 6 members.

   (b) The number of possible committees with 3 members formed from a club of 5 members.

4. (a) The first 15 in the 6th row.

   (b) The first 5 in the 5th row.

5. (a,b) The left-hand column answers Part (a), the right-hand column Part (b).

{A,B}	{C,D,E,F}
{A,C}	{B,D,E,F}
{A,D}	{B,C,E,F}
{A,E}	{B,C,D,F}
{A,F}	{B,C,D,E}
{B,C}	{A,D,E,F}
{B,D}	{A,C,E,F}
{B,E}	{A,C,D,F}
{B,F}	{A,C,D,E}
{C,D}	{A,B,E,F}
{C,E}	{A,B,D,F}
{C,F}	{A,B,D,E}
{D,E}	{A,B,C,F}
{D,F}	{A,B,C,E}
{E,F}	{A,B,C,D}

(c) By adding 10 and 5 (the "10" is the right-hand entry and the "5" is the right-hand one in the fifth row).

(d) There are 20 of them; the following is an incomplete list: {A,B,C}, {A,B,D}, {A,B,E}, {A,B,F}, {A,C,D}, etc.

(e) Yes; 20 = 10 + 10.

6. 1 7 21 35 35 21 7 1

7. 1 8 28 56 70 56 28 8 1

8. 1 23

9. 57 1

7-2. Permutations

The first exercises should enable the students to recognize that in certain situations order or arrangement of elements in a set is important. While A, B, C is *one* selection of three elements, these three elements may be ordered or arranged in *six* different ways. Again the students should be urged to write out all of these arrangements or permutations. It is not too tedious to write out all of the arrangements of 4 elements but, beyond four, time spent in making such lists would be better used in other ways. Particularly the experience of listing the 24 permutations of 4 letters should convince the student of the value in being systematic in constructing the list. However, no particular systematic method, as such, is recommended for use.

In this section the development of formulas is begun. These formulas in themselves are not so important as a tool for problem solving as is the process of developing the formulas and the experience in using symbols, such as, n, (n - 1), and (n - r), to express generalizations.

It may be pointed out that one finds the symbol "${}_nP_r$" as well as the symbol "$P_{n,r}$" in mathematical literature to represent the number of permutations of n elements of a set taken r at a time.

In permutation problems, whether or not an element of a set can be repeated in arrangements must be known. The formulas, of course, apply to permutations without repetitions.

One of the interesting applications is in counting the automobile license plates which can be made by certain rules of selection and arrangements for 10 digits and/or 26 letters. For example, in 1960 in the state of Illinois more than 3 million cars were licensed. The license plates displayed numerals of seven digits. What new method will provide for license plates for 4,000,000 cars and yet involve fewer digits and letters on a given license plate? At the close of this section teachers might encourage individual students to submit plans that Illinois might use and ask the class to discuss the advantages and disadvantages of several of the plans.

Teachers and students will observe that the Multiplication Property has been used before it is introduced. This order was chosen so that the property would be better accepted by students as reasonable and important.

Answers to Class Exercises 7-2a

1. 2

		Treasurer	Secretary
2.	First way	B	E
	Second way	E	B

3. 2

4. 2

5. 6

Answers to Exercises 7-2a

1. (a) Chairman A, Secretary F, Treasurer H;
Chairman A, Secretary H, Treasurer F;
Chairman F, Secretary H, Treasurer A;
Chairman F, Secretary A, Treasurer H;
Chairman H, Secretary A, Treasurer F;
Chairman H, Secretary F, Treasurer A.

(b) 6

2. (a)

Runners

1	2	3	4
P	R	S	T
P	R	T	S
P	S	R	T
P	S	T	R
P	T	R	S
P	T	S	R

Runners

1	2	3	4
R	P	S	T
R	P	T	S
R	S	P	T
R	S	T	P
R	T	S	P
R	T	P	S

Runners

1	2	3	4
S	P	R	T
S	P	T	R
S	R	P	T
S	R	T	P
S	T	P	R
S	T	R	P

Runners

1	2	3	4
T	P	R	S
T	P	S	R
T	R	P	S
T	R	S	P
T	S	R	P
T	S	P	R

(b) 24

3. 6

4.

Presents

1	2	3
A	B	C
A	C	B

Presents

1	2	3
B	A	C
B	C	A

Presents

1	2	3
C	A	B
C	B	A

In 6 different ways.

5. 24

6. 24

7. 23. There is only one correct way. No.

Answers to Exercises 7-2b

1. (a) 3
   (b) 2
   (c) $3 \cdot 2$

2. $9 \cdot 8$

3. $26 \cdot 25$

4. $4 \cdot 3 \cdot 2 \cdot 1$

5. (a) $9 \cdot 8 \cdot 7$
   (b) $9 \cdot 8 \cdot 7 \cdot 6$
   (c) $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

6. $15 \cdot 14 \cdot 13 \cdot 12$

7. (a) n
   (b) $n - 1$
   (c) Yes. There will always be one less chair for the second person to choose because one chair is filled with the first person.
   (d) $n \cdot (n - 1)$
   (e) $n \cdot (n - 1) \cdot (n - 2)$
   (f) $n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3)$

Answers to Exercises 7-2c

1. (a) $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
   (b) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
   (c) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
   (d) $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

2. (a) $7 \cdot (6!)$
   (b) $6 \cdot (5!)$
   (c) $10 \cdot (9!)$
   (d) $12 \cdot (11!)$

3. $\frac{14!}{13!} = \frac{14 \cdot (13!)}{13!} = 14$

4. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6 \cdot 5 \cdot (4!)$

5. $8!$

6. $62! = 62 \cdot 61 \cdot 60 \cdot 59 \cdot \ldots 3 \cdot 2 \cdot 1$
$60! = 60 \cdot 59 \cdot \ldots 3 \cdot 2 \cdot 1$
$\therefore\ 62! = 62 \cdot 61 \cdot (60!)$

7. $9!$ or 362,880

8. $8!$ or 40,320

9. $7!$ or 5,040

10. $8!$ or 40,320

Answers to Exercises 7-2d

1. $7 \cdot 4 = 28$

2. $5 \cdot 3 = 15$

3. $24 \cdot 23 = 552$

4. $50 \cdot 49 = 2,450$

5. $6 \cdot 5 = 30$

6. $26 \cdot 9 \cdot 10 \cdot 10 = 23,400$

7. $5 \cdot 4 \cdot 3 = 60$

8. $26 \cdot 26 \cdot 9 \cdot 10 = 676 \cdot 90 = 60,840$

9. $9 \cdot 10 \cdot 10 \cdot 10 = 9,000$

10. $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 10 \cdot 72 \cdot 42$
$= 30,240$

Answers to Exercises 7-2e

1. $P_{49,7}$

2. (a) $12 \cdot 11 \cdot 10$

   (b) $(12 \cdot 11 \cdot 10) \cdot (9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

   (c) $(12 \cdot 11 \cdot 10) \cdot (9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 12! = P_{12,12}$

3. $P_{12,13} = \frac{12!}{9!}$

4. (a) $20 \cdot 19 \cdot 18 \cdot 17$

   (b) $(20 \cdot 19 \cdot 18 \cdot 17) \cdot (16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

   (c) $20!$ or $P_{20,20}$

5. $P_{20,4} = \frac{20!}{16!}$

6. $P_{n,r} = n(n - 1) \cdot (n - 2) \ldots (n - r + 1)$. (Note that the number of factors here is $n - (n - r + 1) + 1 = r$.)

   (a) Use $(n - r)!$

   (b) Hence, $P_{n,r} = \frac{n!}{(n - r)!}$

7. (a) $26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$ (b) $26!$

   (c) $\frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22}{60 \cdot 60 \cdot 24} = \frac{13 \cdot 23 \cdot 11}{36} = \frac{3289}{36} = 91\frac{13}{36}$ days.

8. (a) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = (P_{10,5})$

   (b) 10^5

9. $5 \cdot 4 = 20$

10. $P_{n,4} > 1600$. Smallest n is 8. $P_{8,4} = 1,680$

7-3. Selections or Combinations

The students should now be ready to develop formulas for use in determining the number of possible selections of n objects taken r at a time. Their work with Pascal's triangle in 7-1 and with permutations in 7-2 is of course very important for the development in this section. We use the notation, $\binom{n}{r}$, which is most widely used in current mathematical literature. However, it is well also for the teacher to call attention to other symbols used for the same ideas, such as, "$C_{n,r}$" and "${}_nC_r$."

It may be desirable to give several more examples, in special cases, of the formula

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdot\ldots\cdot(\text{to } r \text{ factors})}{r!}$$

before attempting to develop it in the general case. In some classes the teacher may choose to designate the r-th factor of the numerator as $(n - r + 1)$. This should be done only if students can understand why the particular choice is made. In practice more people count r factors in the numerator than think what the last factor is by using $(n - r + 1)$.

Students who have completed this chapter with little difficulty and considerable interest should be encouraged to consider additional generalizations about Pascal's triangle, some of which are suggested in the next to the last paragraph of this Commentary, Section 7-1. Another problem in which some students may be interested is associated with the sequence of triangular numbers.

The dots in the following designs are arranged in equilateral triangular fashion.

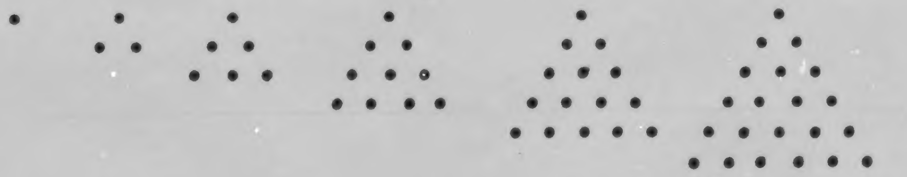
(a) How many dots are there in each design shown? (1, 3, 6, 10, 15, 21)

(b) How are the numbers in the answer to (a) related to the Pascal Triangle? (Third entry in each row.)

(c) If more designs of the same type and of successively larger size were drawn, how many dots would be in the eightieth diagram? (3240)

(d) How many dots in the thousandth diagram? (500,500)

(e) In what sport is the fourth diagram used? (Bowling)



The numbers 1, 3, 6, 10, 15, 21, ... are called triangular numbers. See the Supplementary Unit on Finite Differences.

The ability of the mathematician to predict the future with a maximum degree of certainty is another way in which the power of mathematics may be demonstrated. The formula $\frac{n(n+1)}{2}$ may be used by the student to make a prediction as to how many dots will have to be made to construct the one-thousandth diagram similar to those associated with the sequence of triangular numbers. This activity, while seemingly of little significance, is parallel to the work of the mathematician who predicts that an airplane, still on the drawing boards, will, five years hence, carry a payload of five tons at a speed of 1650 miles per hour at an altitude of 50,000 feet and, at the end of the five-year period be shown correct in every detail. Mathematics, indeed, might be referred to as a "crystal ball"!

[pages 303-308]

Answers to Exercises 7-3a

1. (a) $\binom{12}{7}$ (c) $\binom{m}{3}$

   (b) $P_{12,7}$ (d) $\binom{n+2}{k}$

2. The number of combinations of 6 objects taken 2 at a time; the number of permutations of 8 objects taken 4 at a time; fifty-two factorial; the number of combinations of 52 things taken 13 at a time; the number of permutations of 9 things taken 7 at a time; eight-fourths.

3. (a) 15 and 15 (d) 3 and 3

   (b) 35 and 35 (e) 28 and 28

   (c) 56 and 56

4. $\binom{S}{a} = \binom{S}{b}$. The discussion in Section 1 concerning the number of committees determined by inclusion and exclusion should help to explain the result.

5. (a) 1, 1, 1, 1 (b) $\binom{n}{n} = 1$

6. (a) 1, 1, 1, 1 (b) $\binom{n}{o} = 1$

7. The product $n(n - 1)$ is the number of ways that n things can be arranged 2 at a time. This includes all pairs (AB, AC, AD, ..., BA, CA, DA, ...) and distinguishes between AB and BA, AC and CA, etc. Hence, each of the possible selections is counted twice. Thus, the number of distinct selections is $\frac{n(n - 1)}{2}$.

Answers to Exercises 7-3b

1. $\binom{10}{3} = \frac{10!}{3!7!} = \frac{10\cdot9\cdot8}{3\cdot2\cdot1} = 120$

2. $\binom{15}{5} = \frac{15\cdot14\cdot13\cdot12\cdot11}{5\cdot4\cdot3\cdot2\cdot1} = 3003$

3. $\binom{8}{3} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1} = 56$

4. (a) $\binom{12}{2} = \frac{12\cdot11}{2\cdot1} = 66$

   (b) Twice as many, $132 = P_{12,2}$

5. $4\cdot3\cdot5 = 60$

6. 10 weeks and 2 days more. $(\frac{4\cdot6\cdot3}{7})$.

7. $\binom{52}{13}$ or $\frac{52!}{13!39!}$ or $\frac{52\cdot51\cdot50\cdot49\cdot48\cdot47\cdot46\cdot45\cdot44\cdot43\cdot42\cdot41\cdot40}{13\cdot12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$

8. $8! = 40{,}320$

9. $5\cdot\binom{8}{2} = 5\cdot28 = 140$

10. $\binom{8}{1} + \binom{8}{2} = 8 + 28 = 36$

11. 63. Note that $2^6 - 1 = 63$, and that

    $\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 63$.

7-4. Review of Permutations and Selections

Answers to Exercises 7-4

1. $P_{6,6}$ or $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or $6!$, or 720 words.
2. $2 \cdot 3$ or 6 outfits.
3. $\binom{96}{2}$ or $\frac{96 \cdot 95}{2}$ or 4560 hand-shakes
4. $3 \cdot 7 \cdot 5$ or 105 cars.
5. $P_{5,5}$ or $5!$ or 120 orders.
6. $C_{14,5}$ or $\frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ or 2002 committees.
7. $P_{8,3}$ or $8 \cdot 7 \cdot 6$ or 336 signals.
8. $26 \cdot 26$ or 676 call letters.
9. $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10$ or 90,000 license plates.
10. Permutations. $P_{8,5}$ or $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$, or 6720 patterns.
11. Selections: $\binom{8}{5}$ or $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ or 56 sets of 5 books each.
12. (a) 10 (b) 10

Sample Questions

(Answers are at the end.)

1. Fill in with the proper number.

   (a) $\binom{6}{2} = \binom{6}{?}$

   (b) $\binom{?}{1} = \binom{8}{7}$

   (c) $7 \cdot 6! = ?\,!$

   (d) $\frac{10!}{7!} = P_{?,?}$

   (e) $P_{7,4} = \frac{7!}{?!}$

   (f) $\binom{7}{4} = \frac{P_{7,4}}{?\,!}$

   (g) $\binom{250}{248} = \binom{250}{?}$

2. In how many ways can six students be seated in six chairs?

3. What is the second number in the 14th row of the Pascal Triangle as it was shown in this chapter?

4. On a certain railway there are 15 stations. How many different kinds of tickets are needed to provide tickets between any two of the stations, if the same kind of ticket is good in either direction?

5. In a baseball league of 10 teams, each team plays 17 games with each of the others. How many games are played each season?

6. How many committees with 4 members can be formed from a group of nine men?

7. How many numbers are there in the 13th row of the Pascal Triangle as shown in this chapter?

8. If there are 12 members in a club, how many members will be on the committee that can be made up in the largest number of ways? How many ways can this be done?

9. In how many ways can the first three places in a race be decided if there are 8 dogs in the race? (No ties take place.)

10. Four students are choosing among 8 colleges. In how many ways can the students choose a college to attend?

Answers to Sample Questions

1. (a) 4 (e) 3
   (b) 8 (f) 4
   (c) 7 (g) 2
   (d) 10, 3

2. $6! = 720$ 3. 14

4. $\frac{15 \cdot 14}{1 \cdot 2} = 105$

5. $17 \cdot \binom{10}{2} = 17 \cdot \frac{10 \cdot 9}{1 \cdot 2} = 17 \cdot 45 = 765$

6. $\frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 126$ 7. 14

8. 6; 924

9. $P_{8,3} = 8 \cdot 7 \cdot 6 = 336$

10. $8 \cdot 8 \cdot 8 \cdot 8 = 4{,}096$

Chapter 8

PROBABILITY

Nine or ten teaching days are recommended for this chapter.

8-1. Chance Events

Those responsible for the preparation of this textbook believe that the elementary notions of permutations, selections, and probability are an important part of the general education of all junior high school students. In each of the sections and especially in Section 8-2 on empirical probability an effort has been made to give the student some notion of the great and increasing importance of probability in other disciplines, both in science and in the social sciences. The teacher may choose to add to this discussion in class.

Consideration of the importance of probability serves as a motivating device if not extended too far. The students, however, probably will have more interest in actually working problems than in a lengthy treatment of the significance of probability. It is preferable that little emphasis be placed on use of formulas. The problem situations should be simple enough for the student to analyze each situation and determine the probability by counting possible outcomes directly and working with the "samples" of a total population where data on a sample only is available. More will be accomplished by the student if he understands the basic idea than if he is merely able to categorize situations and apply an appropriate formula.

In some situations students may be encouraged to construct "tables" to assist them in counting. It is important that the students understand that each outcome listed in such a table is considered equally likely to occur. Suggestions for easy or systematic ways to tabulate possible outcomes in tossing coins are given in the text. A particular method is not important but the development of a systematic way of tabulating is important. If students are encouraged to make their own models, such as cubes, circles with spinning points, and so on, for experimenting, they should understand that such models are not necessarily "honest." Whenever a model is referred to in an example or problem, it is always assumed to be honest.

The notation P(A) is introduced. Whenever possible this notation should be used so that the students gain familiarity with it. It is important that the teacher utilize correct terminology to instill in the students the value of doing so. Teachers should not, however, insist on exacting notations or terminology if by doing so the student is distracted from an important idea.

The notion that the "law of averages" affects outcomes may be difficult to dispel. Burton W. Jones's book, Elementary Concepts of Mathematics, pages 209-210, is an excellent teacher reference on this matter.

It was felt that the use of dice should be left to the teacher's discretion. A cube with faces, numbered 1, 2, 3, 4, 5, and 6 or lettered A, B, C, D, E, and F serves the purpose equally well. Different colors applied to each face may also be used. It may be that "tossing coins" will be met with some objections. Disks with different colored sides may be used instead of coins. It may be stressed that games of chance are used only because they serve as good models for discussing elementary probability. Historically, interest in gambling led to the first interest in probability.

The property that there are 2^n possible outcomes if a coin is tossed n times may be developed inductively and is interesting as well as useful. Meaning is added to the property if other similar situations are considered, such as the possible outcomes if a cube is tossed n times.

In answering most probability questions related to this property, the answer is the same whether the coins (or the cubes) are tossed n times or in groups of n. Some class discussion of this would be appropriate. One group of pupils might be asked to solve some of the problems, thinking of a coin being tossed n times, while other pupils think of tossing n coins at one time. Results could then be compared.

It is very important that students understand the limitations of probability, particularly in the elementary and somewhat over-simplified situations in which we are using it. The paragraphs in the text referring to the limitations should be carefully discussed in class. Two points cannot be overemphasized. If 15 coins are tossed in succession and by rare chance the first 14 show heads, still the probability that the fifteenth coin will show heads is $\frac{1}{2}$. Note that this question is quite different from the probability of obtaining 15 heads if 15 coins are tossed, which is $(\frac{1}{2})^{15}$ The second point of caution is that if 100,000,000 coins are tossed, or even more, we can never be sure that one-half of them will be heads.

Finally, because these elementary topics used to introduce probability are so closely associated with their applications, students are apt to think of probability as being the applications (tossing coins, selecting marbles, drawing names or numbers) rather than as a part of mathematics used to describe these games, and social or business situations. Probability is a very important branch of modern mathematics. Experts in probability are called probabilists (pronounced prob' abilists). The rewards for probabilists today in research and in teaching are very great indeed. Probability is an excellent field of work for a young scientist of promise to choose.

[pages 311-314]

Answers to Exercises 8-1a

1. $\frac{2}{3}$

2. $\frac{1}{3}$

3. (a) No. The probability of getting a head to show is $\frac{1}{2}$. We should expect 4, 5, or 6 heads to show in 9 tosses.

   (b) $\frac{1}{2}$

   (c) No. The probability is always $\frac{1}{2}$.

4. (a) $\frac{15}{25}$ or $\frac{3}{5}$

   (b) $\frac{1}{25}$

5. $\frac{40}{48}$ or $\frac{5}{6}$

6. (a) No

   (b) Zero

7. (a) $\frac{1}{2}$, $\frac{1}{2}$

   (b) $\frac{1}{2}$

8. (a) $\frac{1}{2}$

   (b) If $P < \frac{1}{2}$ there is not a good chance for the event to occur. It is not very likely to occur.

9. There are 10 prime numbers from 1 to 30.
   (2,3,5,7,11,13,17,19,23,29) $P = \frac{10}{30}$ or $\frac{1}{3}$.

10. Label the hats Y_1, Y_2, F. The possible outcomes are:

    Y_1 Y_2

    Y_1 F　　　$P = \frac{2}{3}$

    Y_2 F

11. (a) $\frac{1}{5}$

(b) $\frac{1}{4}$

(c) No. The total number of possible outcomes is different in each case.

(d) $\frac{1}{3}$

(e) The measure of chance is increasing.

Answers to Exercises 8-1b

1. $\frac{1}{8}$

2. $\frac{3}{8}$

3. $T = S^N$
   $T = 2^5$
   $T = 32$

4. $\frac{5}{35}$ or $\frac{1}{7}$

5. (a) $\frac{1}{2}$
   (b) about 25

6. (a) $\frac{5}{10}$ or $\frac{1}{2}$
   (b) $\frac{3}{9}$ or $\frac{1}{3}$
   (c) $\frac{2}{8}$ or $\frac{1}{4}$

7. (a) 6
   (b) 6^2 or 36
   (c) $\frac{1}{6}$
   (d) $\frac{1}{36}$

8. (a) 4
   (b) $\frac{1}{4}$
   (c) 4^2 or 16
   (d) 4^3 or 64

9. 4 coins 1(H,H,H,H) 4(H,H,H,T) 6(H,H,T,T) 4(H,T,T,T)
1(T,T,T,T)

5 coins 1(H,H,H,H,H) 5(H,H,H,H,T) 10(H,H,H,T,T)
10(H,H,T,T,T) 5(H,T,T,T,T) 1(T,T,T,T,T)

Yes. The number of each possibility seems to fit the pattern of numbers seen in Pascal's triangle.

10. Six possibilities for two heads and 16 total possibilities. $P = \frac{6}{16}$ or $\frac{3}{8}$.

11\.

Possible outcome		Probability
Head(s)	Tail(s)	
5		$\frac{1}{32}$
4	1	$\frac{5}{32}$
3	2	$\frac{10}{32}$
2	3	$\frac{10}{32}$
1	4	$\frac{5}{32}$
	5	$\frac{1}{32}$
Total Probabilities		$\frac{32}{32} = 1$

12. Either three heads and two tails, or two heads and three tails. There are more possible ways of selecting these combinations.

13. $\frac{6}{64}$ or $\frac{3}{32}$

8-2. Empirical Probability

According to the dictionary "empirical" means depending on experience or observation alone. This is a fairly satisfactory definition for the use of the word in the title of Section 8-2. The term is not defined in the student text with the thought that the examples would make the meaning of the term more clear than a definition would.

While it is difficult to give realistic and correct applications of empirical probability, this branch of the subject is so important that certain risks seem worth taking. The reliability of results depend in great part on statistical theory and sampling techniques, that cannot be treated at this level. This should be made entirely clear to the students. Actually, there is less danger in empirical probability of the student failing to understand the limitations. Students should have little difficulty in appreciating the care necessary in interpreting measures of chance in relation to weather predictions or the prediction that a baseball player will make a hit the next time at bat.

Perhaps the applications of probability to weather forecasting in the text and in Problem 11 of this section are the most unreliable of all. Just because a weather forecaster is accurate in 7 out of 10 predictions, we cannot feel at all sure that he will be accurate 7 times in the next ten forecasts. But the very uncertainty of this is one reason why the weather problem is a good introductory example. Note the difference between the types of questions asked and conclusions drawn in the example in the text and in Problem 11. In one case we talk about the reliability of a forecaster and in the other a measure of chance that it will rain or the sun will shine on a certain date.

More and more measures of reliability of manufactured articles are a matter of great concern in industry. While the example given to illustrate this is greatly oversimplified, a notion of the importance of probability theory to industry should be included in the general education of all students. In Problem 4(b), the student will find that there is a reasonable chance that one of the 40 pencil sharpeners may not be satisfactory. Of course the school, as purchaser, would not be happy about this. If only 489 of the 500 sharpeners meet all test requirements the manufacturer will have to make some adjustment in the manufacturing processes.

In Problem 6 life expectancy tables are introduced. Although these datas give more complete information than it is possible to give in other applications on which some very widely used empirical probabilities are based, even this information is severely limited and at the same time more dependent upon sampling theories than a description of the table and its construction implies. Government regulatory agencies protect the consumer against unfair rates which might be based on out-of-date tables. If the teacher wishes to take the time he might ask a question about the possible advantage to a consumer in buying a life insurance policy (on which the rewards are made upon death) or a life annuity (on which the rewards are made during life) when the cost to the consumer is determined by an out-of-date mortality table (out of date because of advances in medical science).

It is suggested that a few of the problems in Exercises 8-2 might be used for class discussion purposes.

Answers to Exercises 8-2

1. (a) $\frac{152}{1600}$ or $\frac{76}{800}$ or $\frac{19}{200}$ or 0.095

   (b) $\frac{19}{200} \cdot 2000 = 190$

2. $\frac{333}{1000}$

3. Probability has been empirically determined to be $\frac{89}{120}$.

4. (a) $\frac{489}{500}$

   (b) No. About 2% of the sharpeners will not be satisfactory. With 40 sharpeners there is some chance that one may be faulty.

5. Insurance companies keep careful records on accidents with related information on age and sex of drivers. These records show that male drivers under 25 years of age are more apt to be involved in accidents then are male drivers over 25 years of age or female drivers of any age. Account would have to be taken of the percent of an age group by sexes that drive cars, the number of drivers who had accidents, and the number of drivers that were not involved in accidents in a given period of time.

6. (a) 69,517

   (b) none

   (c) It should increase the number alive at each age and decrease the number dying during each year.

7. (a) $P = \frac{92,588}{97,978}$ or 0.95 (b) $P = \frac{35,837}{97,978}$ or 0.37

8. (b) Not for setting insurance rates. Some of the data would be obsolete.

   *(c) Take the results of studying many people at various ages for shorter lengths of time.

9. (a) $\frac{1}{100,000}$ or 0.00001 (b) $P = \frac{69,517}{78,653}$ or 0.88

10. The man at age 60. His chance of living another 10 years is less than the others, so the insurance company would have to charge him more for the extra risk the company would take.

11. $$\left.\begin{array}{l} P(A) = \frac{7}{20} \\ P(B) = \frac{11}{20} \end{array}\right\}$$ Sunshine seems more likely. Note that on cloudy days with no rain both statements are false.

8-3. Probability of A or B

In this brief introduction to probability it seems better not to emphasize formulas and not to go far in categorizing types of problems. However, it may be helpful to students to consider at first separately a few "or" and "and" questions. The student should be encouraged to depend in considerable part on his intuition and good sense. For example, it should seem entirely reasonable to him that the probability of drawing a red or a blue marble from a hat containing red, blue, and white marbles (event A or B) is greater than the probability of his drawing a red marble (event A) or of drawing a blue marble (event B). Furthermore, since one adds the number of red and the number of blue marbles to obtain the number of favorable possible outcomes for event A or B, it should appear reasonable to add $P(A)$ and $P(B)$ to obtain $P(A \text{ or } B)$. Some confusion between the addition in $P(A) + P(B)$ and the "or" in $P(A \text{ or } B)$ may result for those who depend on rules and rote learning rather than understanding. This will be particularly true when $P(A \text{ and } B)$ is introduced in Section 8-4.

That $P(A \text{ or } B) = P(A) + P(B)$ only if A and B are mutually exclusive events is, of course, extremely important. However, this criterion has not been emphasized since some students are apt to spend all their time in trying to decide if events are mutually exclusive so that they can apply a certain formula, rather than in trying to think through the actual problem situation. In examining the statements in Problem 8 to be classified as mutually exclusive events or not the teacher will note that it appears

easier to obtain examples of mutually exclusive events than events not mutually exclusive. In the student text, in most of the problems given, the student is asked to find $P(A \text{ or } B)$ where A and B are mutually exclusive. The teacher may wish to discuss problems in class where A and B are not mutually exclusive and to assign these for outside work.
For example, one might ask:

1. If a pointer spins freely on a dial in which the numbers from 1 through 7 are equally spaced, what is the probability of getting a number divisible by 3 or a number divisible by 2? (Answer: $\frac{4}{7}$, since 2, 3, 4, and 6 are numbers divisible by 2 or 3.)

2. In a class there are 14 boys and 10 girls. In the class 3 of the students (2 boys and 1 girl) have red hair. If a student is chosen at random, what is the probability of choosing a boy or a student with red hair? (Answer: $\frac{15}{24} = \frac{5}{8}$. Fifteen of the 24 pupils are boys or students with red hair.

The teacher may wish to encourage students to solve a number of the problems in this section by two methods, when the events are mutually exclusive, by finding the ratio of the number of successful outcomes to the total number of outcomes or by adding probabilities.

In an elementary and short treatment of probability, in which an informal approach is used, it has seemed better not to introduce the set concept and the union and intersection sets. For teachers who wish to take a more formal approach by utilizing what students know about sets and their unions and intersections, Chapters 4 and 5 of *Introductory Probability and Statistical Inference*, revised preliminary edition of an experimental course, prepared for the Commission on Mathematics, College Entrance Examination Board, are strongly recommended as references. This entire book is also recommended for study in preparation for teaching this chapter.

[pages 327-331]

Answers to Exercises 8-3

1. $\frac{4}{8}$ to $\frac{1}{2}$

2. (a) $\frac{2}{6}$ or $\frac{1}{3}$ (b) $\frac{4}{6}$ or $\frac{2}{3}$

3. (a) 1, yes

   (b) Yes. Subtract $\frac{1}{3}$ from 1 to obtain $\frac{2}{3}$.

4. $\frac{8}{10}$ or $\frac{4}{5}$

5. $P(A) + P(B) = 1$

6. (a) $P(\text{red}) = \frac{4}{9}$

   (b) $P(\text{white}) = \frac{3}{9}$ or $\frac{1}{3}$

   (c) $P(\text{red or white}) = \frac{7}{9}$

7. (a) $P(\text{dog or cat}) = \frac{18}{27}$ or $\frac{2}{3}$

   (b) $P(\text{no four-legged animals}) = \frac{3}{27}$ or $\frac{1}{9}$

8. (a) Yes

   (b) Yes

   (c) No

   (d) Yes

   (e) No

   (f) Yes

   (g) Yes

   (h) Yes

   (i) No (Mother may be teacher.)

   (j) Yes.

9. $P(\text{even number}) = \frac{4}{6}$ or $\frac{2}{3}$

10. (a) $\binom{9}{2} = \frac{9 \cdot 8}{1 \cdot 2} = 36$

    (b) $\binom{4}{2} = \frac{4 \cdot 3}{1 \cdot 2} = 6$

    (c) $\binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 2} = 10$

(d) Red card and a black card = 20.

(e) The sum of (b), (c), and (d) is the same as (a).

11. (a) $\frac{10}{36} + \frac{6}{36} = \frac{16}{36}$ or $\frac{4}{9}$ (b) $\frac{20}{36}$ or $\frac{5}{9}$

12. (a) Two coins at the same time.

1st coin	2nd coin
H	H
H	T
T	H
T	T

Three out of four possibilities have at least one coin with a head.

(b) Yes. One in each hand.

(c) When they are tossed separately both events can happen. P(A) + P(B) = P(C) only when the two events are mutually exclusive.

8-4. Probability of A and B

Again it is recommended that the treatment of this new situation be informal and that the student be encouraged to "think through" a problem rather than to use a formula. It should be clear to him that the probability that event A and event B both occur is less than P(A) and less than P(B). For example, if the probability that Joe will get an A in mathematics is 0.9 and the probability that he will get an A in French is 0.7 the probability that he will get an A in both subjects is less than that he will get at least one A.

A point of distinction between P(A or B) and P(A and B) is that in the former, where A and B are mutually exclusive, we refer to a single toss of a coin, a single spin of a wheel, or a single selection, while in the latter at least 2 tosses of a coin, 2 spins of a pointer, or 2 selections are involved.

[pages 333-336]

With a few examples we ask the student to conclude that if events A and B are independent,

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

There is, of course, no proof of this property in the text. It is important that the student understand that P(A and B) is the product of P(A) and P(B) only if A and B are independent. Events A and B are independent only if the occurrence of A or the non-occurrence of A have no effect whatsoever on B. An example of two events which are not independent might be the selection of two marbles from a box containing n marbles, in which the first marble selected is not replaced before the second selection or drawing.

Again the material is presented in a way intended to encourage the student to think through a problem situation and to use his best judgment rather than to apply a formula.

Answers to Exercises 8-4

1. (a) Yes. One event has no effect on the other.
   (b) $\frac{1}{4}$

2. (a) If A,B,C and D are independent events, then
   $P(A \text{ and } B \text{ and } C \text{ and } D) = P(A) \times P(B) \times P(C) \times P(D)$.
   (b) $(\frac{1}{2})^9 = \frac{1}{512}$.

3. (a) $\frac{2}{10}$ or $\frac{1}{5}$
   (b) $\frac{2}{5}$ (assuming A was played first.)

4. (a) P(both on red) = $\frac{2}{4} \cdot \frac{3}{6} = \frac{1}{4}$

   (b) P(both on green) = $\frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}$

   (c) P(white and blue) = $\frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}$

5. P(white and white) = $\frac{4}{9} \cdot \frac{4}{9} = \frac{16}{81}$

6. $\frac{4}{9} \cdot \frac{3}{8} = \frac{1}{6}$. With combinations or selections:

   $\frac{\binom{4}{2}}{\binom{9}{2}} = \frac{6}{36} = \frac{1}{6}$ where $\binom{4}{2}$ is number of pairs of two whites

   and $\binom{9}{2}$ is the number of possible pairs.

7. (a) Yes (b) No

8. (a) $\frac{1}{2}$ (c) $\frac{1}{4}$

   (b) $\frac{1}{4}$ (d) $\frac{1}{2}$

9. (a) Not independent (d) Independent

   (b) Independent (e) Independent

   (c) Not independent

10. Event A: The first man will solve the problem.

    Event B: The second man will solve the problem.

    (a) Neither man can solve the problem.

    P(not A and not B) = $\frac{1}{3} \cdot \frac{7}{12} = \frac{7}{36}$.

(b) Either man or both men may solve the problem.

$$P(A \text{ and not } B) = \frac{2}{3} \cdot \frac{7}{12} = \frac{14}{36}$$

$$P(\text{not } A \text{ and } B) = \frac{1}{3} \cdot \frac{5}{12} = \frac{5}{36}$$

$$P(A \text{ and } B) = \frac{2}{3} \cdot \frac{5}{12} = \frac{10}{36}$$

Total $\frac{29}{36}$.

This could have been found by $1 - \frac{7}{36} = \frac{29}{36}$.

11. The probability is equal to one (the desired committee may be chosen in one way) divided by the total number of ways a committee of three can be chosen.

$$P = \frac{1}{\binom{20}{3}} = \frac{1}{\frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3}} = \frac{1}{20 \cdot 19 \cdot 3} = \frac{1}{1140}.$$

12. There are 2^6 ways that the six coins may have heads and tails appear. There is only one way that will not have at least one head. (6 tails.) Therefore,

$$P(\text{at least one head}) = \frac{2^6 - 1}{2^6} = \frac{64 - 1}{64} = \frac{63}{64}.$$

13. (a) 1 (c) $\frac{1}{4}$

(b) 0 (d) $\frac{1}{2}$

14. List the possible lengths for a triangle:

10, 9, 8
10, 9, 7
10, 9, 6
10, 9, 5
10, 9, 4
10, 9, 3
10, 9, 2

10, 8, 7
10, 8, 6
10, 8, 5
10, 8, 4
10, 8, 3

10, 7, 6
10, 7, 5
10, 7, 4

10, 6, 5

9, 8, 7
9, 8, 6
9, 8, 5
9, 8, 4
9, 8, 3
9, 8, 2
9, 7, 6
9, 7, 5
9, 7, 4
9, 7, 3
9, 6, 5
9, 6, 4

8, 7, 6
8, 7, 5
8, 7, 4
8, 7, 3
8, 7, 2
8, 6, 5
8, 6, 4
8, 6, 3
8, 5, 4
7, 6, 5
7, 6, 4
7, 6, 3
7, 6, 2
7, 5, 4
7, 5, 3

6, 5, 4
6, 5, 3
6, 5, 2
6, 4, 3
5, 4, 3
5, 4, 2
4, 3, 2

There are 50 such combinations.

$$P = \frac{50}{\binom{10}{3}} = \frac{50}{120} = \frac{5}{12}.$$

15. Method I.

(a) This is a problem where replacements have not been made. Use combinations.

There are $\binom{5}{2}$ ways of drawing even numbers.

There are $\binom{10}{2}$ ways of drawing any kind of pair.

Thus, $P(2 \text{ even numbers}) = \dfrac{\binom{5}{2}}{\binom{10}{2}} = \dfrac{10}{45} = \dfrac{2}{9}$.

Method II.

(a) Using $P(A \text{ and } B) = P(A) \cdot P(B)$.

$P(A) = \frac{1}{2}$; $P(B) = \frac{4}{9}$.

$P(A \text{ and } B) = \frac{1}{2} \cdot \frac{4}{9} = \frac{2}{9}$.

(b) The sum of two even numbers is an even number; therefore the information in (a) is a start. The sum of two odd numbers is also an even number; therefore we need pairs of odd numbers. There are $\binom{5}{2} = 10$ pairs of odd numbers. The probability of drawing two odd numbers is the same as drawing two evens, that is $\frac{2}{9}$.

Since we can satisfy the requirements of the problem either by drawing two even or two odds we add the probabilities.

Thus, P(that the sum of the two numbers is even)

$$= \frac{2}{9} + \frac{2}{9} = \frac{4}{9}.$$

This naturally could have been solved by finding the number of ways of obtaining a sum which was odd.

The P(of obtaining a sum which is odd) = $\frac{25}{45}$.

(Each odd number--there are 5--must be matched with an even number--there are 5. Thus, the numerator is $5 \times 5 = 25$.)

(c) List all the ways that the sum of the two numbers is divisible by 3.

1 and 2
1 and 5
1 and 8
2 and 4
3 and 6
2 and 7
4 and 5
4 and 8
3 and 9
2 and 10
5 and 7
5 and 10
6 and 9
7 and 8
8 and 10

P(the sum is divisible by 3) = $\frac{15}{45} = \frac{1}{3}$.

(d) P(the sum is less than 20) = 1.

(e) P(the pairs have a sum more than 20) = 0.
Note: (d) cannot fail and (e) cannot happen.

16. BRAINBUSTER.

(a) P(one of the pair being a dime) equals the number of pairs with a dime divided by the number of pairs with two heads. There are 3 pairs with a dime--DP, DN, and DQ. There are $\binom{4}{2} = 6$ possible pairs with two heads. ∴ P(one of the pair a dime) = $\frac{3}{6} = \frac{1}{2}$.

(b) P(one of the three heads a dime) equals the number of ways a dime may be a member of the three coins with heads showing divided by the number of ways there can be three heads showing. There are $\binom{3}{2} = 3$ ways that a dime may be one of the coins. (DPN, DPQ, DNQ). There are $\binom{4}{3} = 4$ ways that three heads may show. P(one of three heads a dime) = $\frac{3}{4}$.

17. BRAINBUSTER.

(a) Probability equals the number of ways that (DQ?) may occur with all heads showing divided by the number of ways that (???) may occur with all heads showing.

$$P = \frac{\binom{3}{1}}{\binom{5}{3}} = \frac{3}{10}.$$

(b) Probability equals the number of ways that (D?) may occur with heads divided by the number of ways that (??) may occur with heads showing.

$$P = \frac{\binom{4}{1}}{\binom{5}{2}} = \frac{4}{10} = \frac{2}{5}.$$

(c) P(2 heads and one a dime) = P(2 heads) · P(one of the pair is a dime).

$$P(2 \text{ heads}) = \frac{\binom{5}{2}}{2^5} = \frac{10}{32} = \frac{5}{16}.$$

$$P(\text{one of pair a dime}) = \frac{\binom{4}{1}}{\binom{5}{2}} = \frac{4}{10} = \frac{2}{5}.$$

$$P(2 \text{ heads and one a dime}) = \frac{5}{16} \cdot \frac{2}{5} = \frac{1}{8}.$$

(d) Event A: 3 coins come up heads.

Event B: 2 of these 3 are a dime and a quarter.

P(A and B) = P(A) · P(B).

$$P(A) = \frac{\binom{5}{3}}{2^5} = \frac{10}{32} = \frac{5}{16}.$$

$$P(B) = \frac{\binom{3}{1}}{\binom{5}{3}} = \frac{3}{10}.$$

$$P(A \text{ and } B) = \frac{5}{16} \cdot \frac{3}{10} = \frac{3}{32}.$$

Answers to Exercises 8-5

1. (a) $\frac{4}{16}$, or $\frac{1}{4}$ (d) $\frac{7}{16}$

   (b) $\frac{5}{16}$ (e) $\frac{1}{16}$

   (c) $\frac{6}{16}$, or $\frac{3}{8}$

2. (a) $\frac{12}{17}$ (c) 1

   (b) $\frac{5}{17}$

3. $\frac{1}{32}$

4. (a) $\frac{13}{52}$, or $\frac{1}{4}$ (c) $\frac{1}{52}$

(b) $\frac{4}{52}$, or $\frac{1}{13}$ (c) $\frac{26}{52}$, or $\frac{1}{2}$

5. Total possibilities = $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$, or 720. Probability = $\frac{1}{720}$.

6. $\frac{4}{5}$

7. (a) $\frac{1}{3}$ (b) $\frac{2}{3}$

8. $\frac{1}{3} + \frac{2}{3} = 1$. The sum must be 1 since (3b) is just another way of saying (not 3a), and we know that the probability that an event will occur plus the probability that an event will not occur is 1.

9. $\frac{2}{3}$ if you include the possibility of standing between them.

$\frac{1}{2}$ if you do not include this possibility.

10. (a) $\frac{9}{29}$ (c) $\frac{17}{29}$

(b) $\frac{8}{29}$

11. $(0.313)(0.260)(0.300) = 0.024414 \approx 0.024$

12. (a) $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ (b) $\frac{4}{8} = \frac{1}{2}$

13. $\frac{5}{12}$

14. $\frac{500}{725}$ or $\frac{20}{29}$

15. (a) $0.4 \times 0.6 = 0.24$

(b) $0.6 \times 0.4 = 0.24$

(c) $0.4 \times 0.4 = 0.16$

(d) Panthers lose the Bear's game and win the Nationals' game. $0.6 \times 0.6 = 0.36$.

16. (a) $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$

(b) [✓ 1H], [✓ 2H], [✓ 3H], [✓ 4H], [✓ 5H], [✓ 6H], 1T, 2T, 3T, 4T, [✓ 5T], 6T. $P = \frac{7}{12}$

(c) [✓ 1H], [✓ 2H], [✓ 3H], [✓ 4H], [✓ 5H], [✓ 6H], 1T, [✓ 2T], 3T, [✓ 4T], 5T, [✓ 6T]. $P = \frac{9}{12}$

17. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64$. Some may think of this problem as $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$. You may wish to point out that this is the sixth row of Pascal's triangle.

Sample Questions

True-False

T 1. If the probability of winning the football game next Friday is $\frac{3}{5}$ the team is more likely to win than to lose.

T 2. If a box contains two red and two black marbles, the probability of picking a red marble is the same as the probability of picking a black marble on the first draw.

F 3. If a penny is tossed 20 times, we expect to have heads show exactly 10 times.

F 4. Probability is always a number less than $\frac{15}{16}$.

T 5. If two pennies are tossed 100 times, we can expect two heads to show on about 25 tosses.

T 6. The probability of "X" winning the race is $\frac{3}{7}$. "X" has less than an even chance of winning.

T 7. Decision making can be based on a knowledge of chance.

T 8. A statement, such as, a weather forecast is a chance statement.

F 9. An honest coin is one that when tossed and allowed to fall freely will always show heads.

T 10. Probability is sometimes used by governments.

F 11. If two pennies are tossed 100 times, we can expect that two heads will show on exactly 25 tosses.

F 12. If two coins are tossed, three outcomes are possible: two heads, two tails, a head and a tail. Thus, the probability of tossing two heads is one-third.

T 13. If five coins are tossed, the probability that two are heads is the same as the probability that three are tails.

F 14. Drawing a spade and drawing an ace from a deck of cards are mutually exclusive events.

T 15. If events A and B are independent the probability of the event: A and B, is the product of the probability of A and the probability of B.

Multiple Choice

D 1. Suppose you toss a coin five times and each time it shows heads. The probability that heads will show on the next toss is ...

A. $\frac{5}{6}$

B. $\frac{1}{6}$

C. $\frac{1}{5}$

D. $\frac{1}{2}$

E. None of the above is correct.

A 2. Mark has three cards. One card has the letter A printed on it, another the letter C, and the third has a T. If he mixes (shuffles) the cards with the letters face down and turns them over one at a time, what is the probability that they will be turned over in the order CAT?

A. $\frac{1}{6}$
B. $\frac{1}{3}$
C. $\frac{1}{2}$
D. $\frac{3}{8}$
E. None of the above is correct.

C 3. A weatherman's forecasts are correct 20 times in one month and not correct 10 times in the same month. Based on this information, the estimate of the measure of chance that his future forecasts will be correct is ...

A. $\frac{1}{3}$
B. $\frac{1}{2}$
C. $\frac{2}{3}$
D. $\frac{3}{4}$
E. None of the above is correct.

B 4. A drawer contains 10 blue stockings and 10 red stockings. What is the smallest number of stockings you need to take to be sure you have a pair of the same color stockings?

A. 2
B. 3
C. 5
D. 10
E. 11

A 5. There are two roads from Milltown to Luck and three roads from Luck to Turtle Lake. How many different ways can John travel from Milltown to Turtle Lake?

A. 6
B. 5
C. 3
D. 2
E. None of these.

E 6. If six honest coins are tossed and allowed to fall freely, how many possible outcomes are there?

A. 4
B. 8
C. 16
D. 32
E. None of the above is correct.

D 7. Each face of two regular tetrahedrons is painted a particular color, red, blue, yellow, and green. If the tetrahedrons are rolled what is the chance that both solids will stop on the white faces?

A. $\frac{1}{2}$
B. $\frac{1}{4}$
C. $\frac{1}{8}$
D. $\frac{1}{16}$
E. None of the above is correct.

Computation Questions

Compute the measures of chance for these events.

($\frac{1}{3}$) 8. A box contains a red marble, green marble and yellow marble. If you draw two marbles without looking, what are the chances that you draw the red and green marble in any order?

$(\frac{2}{3})$ 9. Three jackets are in a dark closet. One jacket belongs to you, one to Bill, and one to Bob. If you reach into the closet and take out two jackets, what is the chance that you get your own jacket?

(8) 10. How many equally likely events are possible if a regular octahedron (an eight-sided solid) is rolled?

(64) 11. How many outcomes are possible if two regular octahedrons are rolled simultaneously (at the same time)?

$(\frac{2}{3})$ 12. Three cards are numbered like the ones below.

$$\boxed{3} \quad \boxed{5} \quad \boxed{6}$$

Without looking at the numbers you are to draw any two cards. What is the chance that the sum of the two numbers on the cards you draw is odd?

13. Assume that the probability of a family having a boy is $\frac{1}{2}$, and having a girl is also $\frac{1}{2}$.

$(\frac{1}{8})$ (a) What is the probability of a family having three boys in succession?

$(\frac{7}{8})$ (b) What is the probability in a family with three children of not having three girls?

$(\frac{1}{8})$ (c) What is the probability of a family having two boys and a girl, in that order?

$(\frac{3}{8})$ (d) What is the probability of a family with three children having two boys and a girl, in any order?

$(\frac{3}{8})$ (e) What is the probability of a family with four children having two boys and two girls, in any order?

$(\frac{1}{16})$ (f) What is the probability of a family with four children having four girls?

$(\frac{5}{8})$ (g) What is the probability in a family of four children of not having half boys, half girls?

(one out of 4096). (h) Occasionally one reads in the newspaper of a family that has twelve children, all of them boys. In families that have twelve children, how often is this likely to happen?

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Chapter 9

SIMILAR TRIANGLES AND VARIATION

Introduction

This chapter involves ideas from both algebra and geometry. In the interplay, each contributes to the other. The central geometrical concept here is that of similar triangles, and the fundamental algebraic ideas center around variation and proportion.

These ideas are introduced first by discussion of congruent right triangles and the trigonometric ratios beginning with the development of the idea that each of the ratios $\frac{y}{x}$, $\frac{y}{r}$, $\frac{x}{r}$ depends only on the angle between the X-axis and the ray which connects the point (x,y) with the origin. Then the ratios are defined by name in terms of these letters and finally in terms of the sides of a right triangle. Applications to measurement are given, and here some review of the chapter on relative error and material in the Seventh Grade course may be needed.

After fixing ideas for the right triangle, the concept of similarity of triangles in general is developed. Then follow two applications: slope, which is merely a variant of the trigonometric tangent, and scale drawings and maps which apply and extend the idea of similarity. Then, finally, the connection with direct and inverse variation is the more algebraic phase of the material already covered.

Not only are the various topics in this chapter intimately interrelated but there are many relationships to and applications of previous material: graphs, approximate measurement, right triangles, ratio, equations.

The first part of this chapter has been drastically revised, chiefly in the direction of concentrating first on a careful development of the right triangle ratios before moving into the more general notion of similar triangles.

Recommended teaching time: 15 days.

9-1. Indirect Measurement and Ratios

The object of this section is to develop the idea that the values of the three ratios $\frac{y}{x}$, $\frac{y}{r}$, $\frac{x}{r}$ depend only on the angle between the X-axis and the ray which connects the point (x,y) with the origin and not on the particular point (x,y) on that ray which is chosen. To this end, various examples in terms of coordinates are used.

For Class Exercises 9-1a these suggestions are given. Probably the first five problems should be a cooperative class endeavor. The sixth can be done by each member of the class at his seat and their results tabulated and compared. With some of the remaining problems some members of the class may need much help, but probably the better ones can do most of the work unaided.

Answers to Class Exercise 9-1a

1. (a) $\frac{6}{8}$ (b) $\frac{9}{12}$
2. (a) $\frac{3}{5}$ (b) $\frac{6}{10}$
3. (a) $\frac{4}{5}$ (b) $\frac{8}{10}$

4.

1	2	3
$\frac{3}{4}$	$\frac{3}{5}$	$\frac{4}{5}$
$\frac{6}{8}$	$\frac{6}{10}$	$\frac{8}{10}$
$\frac{9}{12}$	$\frac{9}{15}$	$\frac{12}{15}$
$\frac{12}{16}$	$\frac{12}{20}$	$\frac{16}{20}$

5. (a) $\frac{3}{4}$

(b) Yes, they are equal. $\frac{3}{5}$.

(c) $\frac{4}{5}$. Yes, they are equal.

6. (a) $x = 6$

(b) $\frac{4\frac{1}{2}}{6} = \frac{4.5}{6} = \frac{4.5}{6}\left(\frac{2}{2}\right) = \frac{9}{12} = \frac{3}{4}$.

7. (a) Right triangle. (b) Yes.

8. (a) hypotenuse

(b) $c^2 = a^2 + b^2$ or $r^2 = x^2 + y^2$

$c = \sqrt{a^2 + b^2}$ or $r^2 = \sqrt{x^2 + y^2}$

$r = \sqrt{6^2 + (4.5)^2}$

$r = \sqrt{36 + 20.25}$

$r = \sqrt{56.25}$

$r \approx 7.5$

$ON \approx 7.5$

9.

1	2	3
$\frac{4}{5}$	$\frac{4}{6.4}$	$\frac{5}{6.4}$
$\frac{8}{10}$	$\frac{8}{12.8}$	$\frac{10}{12.8}$
$\frac{12}{15}$	$\frac{12}{19.2}$	$\frac{15}{19.2}$

10. (a) Right triangles

(b) Yes. Because OB, OD, and OF are determined by the same ray $\overrightarrow{OB}$.

(c) $\frac{y}{x}$ is equal for the three triangles; $\frac{4}{5}$.

(d) Yes. They are all equal.

(e) Yes.

11. For Δ AOB

$r = \sqrt{x^2 + y^2}$, similarly, OD $\approx$ 12.8 and OF $\approx$ 19.2.

$r = \sqrt{4^2 + 5^2}$

$r = \sqrt{16 + 25}$

$r = \sqrt{41}$

$r \approx 6.4$

or OB $\approx$ 6.4

Answers to Exercises 9-1a

1. (a) $\frac{y}{x} = \frac{4}{3}$ $\frac{y}{r} = \frac{4}{5}$ $\frac{x}{r} = \frac{3}{5}$

(b) $\frac{y}{x} = \frac{8}{6}$ $\frac{y}{r} = \frac{8}{10}$ $\frac{x}{r} = \frac{6}{10}$

(c) Answers will vary. The ratios should be the same, but these may vary if points whose coordinates are not integers are used. Errors in measurement will affect pupils' answers.

2. (a)

	$\frac{y}{x}$	$\frac{y}{r}$	$\frac{x}{r}$
ΔAOB	$\frac{5.2}{3}$	$\frac{5.2}{6}$	$\frac{3}{6}$
ΔCOD	$\frac{10.4}{6}$	$\frac{10.4}{12}$	$\frac{6}{12}$
ΔEOD	$\frac{15.6}{9}$	$\frac{15.6}{18}$	$\frac{9}{18}$

(b) Yes, they are equal.

(c) Yes, they are equal.

(d) Yes, they are equal.

(e) The results should agree with the table.

3. (a) $\frac{5.2}{3}$

(b) $\frac{y}{x} = T$

$\frac{y}{45} = \frac{5.2}{3}$ (or about 1.73)

$y = \frac{5.2}{3}(45)$ or $(1.73)(45)$

$y \approx 78$ or 77.8

4. (a) $\frac{5.2}{6}$

(b) $\frac{y}{75} = \frac{5.2}{6}$

$y \approx 65$

5. (a) right triangle

   (b) 45° since ΔAOB and ΔCOD are isosceles, right triangles.

   (c) 1

   (d) Should be equal to 1.

   (e) No.

6. In ΔOST, $y = 1$ and $x = 1$.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + 1^2} \\ &= \sqrt{2} \end{aligned}$$

Thus, $\frac{y}{r} = \frac{1}{\sqrt{2}}$

Answers to Class Exercises 9-1b

1. (a) $\overline{AC}$, $\overline{CE}$, $\overline{BR}$, $\overline{DS}$.

   (b) They are congruent by property A.S.A.

   (c) $\overline{DR}$, $\overline{CR}$, $\overline{FS}$.

   (d) 2b

   (e) 3b

2.

ΔAOB	$\frac{b}{a}$	$\frac{b}{c}$	$\frac{a}{c}$
ΔCOD	$\frac{2b}{2a}$	$\frac{2b}{2c}$	$\frac{2a}{2c}$
ΔEOF	$\frac{3b}{3a}$	$\frac{3b}{3c}$	$\frac{3a}{3c}$

3. (a) Yes. Multiplication property of 1.
   (b) Yes. Yes.
   (c) Yes. Yes. Yes.

4. (a) $4b$
   (b) $\frac{4b}{4a} = (\frac{4}{4})\frac{b}{a} = \frac{b}{a}$

5. (a) $\frac{1}{2}b$ or $\frac{b}{2}$
   (b) $\frac{\frac{1}{2}b}{\frac{1}{2}a} = \frac{\frac{1}{2}}{\frac{1}{2}}(\frac{b}{a}) = \frac{b}{a}$

Answers to Exercises 9-1b

1. (a) $\frac{12}{7}$
   (b) $OB \approx 13.9$
   (c) $\frac{12}{13.9}$
   (d) $\frac{7}{13.9}$
   (e) A little less than 60°.

2. (a) $\frac{y}{x} \approx \frac{9.8}{9.8}$ or 1
   (b) $OB \approx 13.9$
   (c) $\frac{y}{r} \approx \frac{9.8}{13.9}$ and $\frac{x}{r} \approx \frac{9.8}{13.9}$. They are not the same as the answers for 1(c) and 1(d).

3. $y \approx 164$ feet. (164.35)

4. $r \approx 91.9$ or 92 feet.

5. 30°

6. (a) About 5 feet. (5.1)
   (b) About 13 feet.
   (c) $13^2 = 12^2 + 5^2$
   $169 = 144 + 25.$

9-2. Trigonometric Ratios

In this section we give the usual names to the ratios dealt with in the previous section. A table for 30°, 45°, 60° is constructed and used. Aside from the construction of this table, the purpose of the Class Exercise 9-2 is to fix in mind the definitions of the trigonometric ratios "forwards and backwards" and to give students some practice in picking out the proper ratio for a given problem.

This is intended only as a very brief introduction to the methods and the simplest problems of trigonometry. The purpose of this section is to fix the ideas of the previous section and to add the interest which computation of this kind induces.

Answers to Class Exercise 9-2

1. (a) 10
   (b) About 17.3
   (c) $\tan \angle 60^\circ \approx 1.73$
   (d) $\tan \angle 45^\circ \approx 1.00$
   $\tan \angle 30^\circ \approx .57$

2. (a) 10
   (b) About 8.7
   (c) $\sin \angle 60^\circ \approx .87$
   (d) $\sin \angle 45^\circ \approx .70$
   $\sin \angle 30^\circ \approx .50$
   (e) About 5
   (f) $\cos \angle 60^\circ \approx .50$
   (g) $\cos \angle 45^\circ \approx .70$
   $\cos \angle 30^\circ \approx .87$

3. (a) They appear to be equal.

   (b) They appear to be equal.

4.

$m(\angle BOA)$	$\sin \angle BOA$	$\tan \angle BOA$	
30	.50	.57	60
45	.70	1.00	45
60	.87	1.73	30
	$\cos \angle BOA$ (Read up)		$m(\angle BOA)$ Read up for cosine only

5. (a) Tangent

   (b) .57

   (c) $\frac{y}{6} = \tan \angle BOA$

   $y = 6(\tan \angle BOA)$

   $y = 6(.57)$

   $y \approx 3.42$

6. (a) Sine (c) Tangent

   (b) Cosine (d) Cosine

7. (a) $\overline{TR}$ (c) $\overline{ST}$

   (b) $\overline{ST}$ (d) $\overline{TR}$

8. (a) $\overline{EF}$ (c) $\overline{ED}$

   (b) $\overline{ED}$ (d) $\overline{EF}$

9. (a) $\overline{LK}$, $\overline{JL}$ (c) $\overline{JK}$

   (b) $\overline{JL}$, $\overline{LK}$

10. (a) To find BA, use the tangent; to find OB, use the cosine.

    (b) To find OA, use the cosine; to find AB, use the sine.

    (c) To find OB, use the sine; to find OA, use the tangent.

11. (a) To find BA, use the tangent; to find OB, use the sine.

    (b) To find OA, use the sine; to find AB, use the cosine.

    (c) To find OB, use the cosine; to find OA, use the tangent.

Answers to Exercises 9-2

1. $\tan 60^\circ = \frac{BP}{AP} = \frac{BP}{50}$. Hence, $50 \tan 60^\circ = BP$.
   Thus, $BP \approx 50(1.73) \approx 86.5$.

2. Let x denote the height it touches on the building. Then $\sin 45^\circ = \frac{x}{25}$. Hence, $x = 25 \sin 45^\circ \approx 25(.70) \approx 17.5$.

3. Since $\cos 45^\circ = \sin 45^\circ$, the answer is the same as in the previous problem.

4. $\tan 30^\circ = \frac{ED}{EF} = \frac{ED}{150}$. Hence, $ED = 150 \cdot \tan 30^\circ \approx 150(0.577)$ ≈ 86.6.

5. (a) $\sin 45^\circ = \frac{4}{\text{length of the hypotenuse}}$. Hence, the length of the hypotenuse is equal to

$$\frac{4}{\sin 45^\circ} \approx \frac{4}{.707} \approx 5.64.$$

(b) The length of the hypotenuse is equal to

$$\sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2} \approx 5.64.$$

6. $\frac{\sin 60^\circ}{\sin 30^\circ} \approx \frac{.866}{.500} \approx 1.73, \quad \frac{\tan 60^\circ}{\tan 30^\circ} \approx \frac{1.732}{0.574} \approx 3.02.$

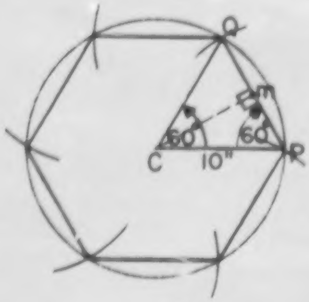
We have seen from previous constructions that when we double the angle we do not double the value of the ratio.

7. (a) $m(\angle PCQ) = 60$

(b) $m(\angle CPQ) = 60$

(c) $CM = \frac{10\sqrt{3}}{2} \approx \frac{10(1.732)}{2} \approx 8.66$

(d) 10

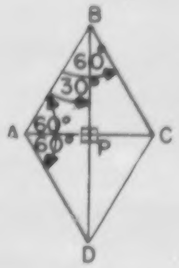


8. (a) $m(\angle ABD) = 30$

(b) $m(\angle DBC) = 30$

(c) $\overline{BD} \perp \overline{AC}$ because $m(\angle APB) = 90$

(d) $BD \approx 17.3$



9-3. Reading a Table

It is important that pupils learn to read a mathematical table. They have already had some experience in doing this (see table of squares and square roots of numbers in Chapter 4). The process of reading a table does not cause them much difficulty but the special construction of the table of trigonometric ratios makes careful instruction here necessary. An important by-product is the emphasis on the co-function relationship and the definition of the ratio called the cotangent.

The approximate nature of the numbers, correct to ten-thousandths to be found in the table should be emphasized. It might be pointed out that this table was not computed from measurements (although with modern instruments this would be possible) but that more advanced mathematics gives other ways of constructing such a table.

Notice that angle measurements 0° and 90° are avoided. The teacher is here referred to the Teachers' Commentary on Chapter 4, Section 5. The reason for avoiding these here is the conflict between the definition in terms of right triangles, which here breaks down, and the definition in terms of coordinates of points on rays which not only applies in defining trigonometric functions of 0° and 90° but obtuse angles and more general angles as well.

Some students, however, may ask what are the values of the functions for 0° and 90° and the teacher may wish to be prepared to answer. For an angle of 0°, the ray which we have called $\overrightarrow{OB}$ lies along the X-axis and if we choose the point B to be (1,0) we have $y = 0$, $x = 1$, $r = 1$ and, hence,

$$\sin 0^\circ = \frac{0}{1} = 0, \quad \cos 0^\circ = \frac{1}{1} = 1, \quad \tan 0^\circ = \frac{0}{1} = 0.$$

For the angle of 90° the ray will lie along the Y-axis and the point B may be taken to be (0,1). Here $x = 0$, $y = 1$, $r = 1$, and it is easy to find that:

$$\sin 90^\circ = 0, \quad \cos 90^\circ = 1.$$

But the third function, $\tan 90^\circ$, presents special difficulties since the definition would give $\frac{1}{0}$ which is not a number. Thus, there is no number which is equal to $\tan 90^\circ$. In other words, the tangent of a right angle does not exist.

Answers to Exercises 9-3

1. (a) 0.1736
   (b) 0.1763
   (c) 0.6561
   (d) 0.8910
   (e) 0.9903
   (f) 0.8391
   (g) 1.1918
   (h) 1.7321
   (i) 2.7475
   (j) 0.9994

2. (a) yes
   (b) yes
   (c) yes
   (d) yes
   (e) yes

3. (a) The tangent of an angle is greater than zero but not always less than one.

   (b) The tangent of an angle increases with the size of the angle between 1° and 89°.

   (c) The differences between consecutive table readings varies throughout the table.

   (d) The differences between the tangents of two consecutive angles are smaller for smaller consecutive angles than for larger consecutive angles.

4. (a) 52.99
   (b) 89.9586
   (c) 0.2582|01 or 0.2582
   (d) 0.2835|65 or 0.2836

5. $w = 20(\tan 40^\circ) \approx 20(0.8391) \approx 16.782$

6. $\frac{5}{4} = \cot \angle CAB = 1.25$.
   In the table, $\cot 39^\circ \approx 1.2349$ and $\cot 38^\circ \approx 1.2799$.
   Hence, $\cot 1.25 \approx 39^\circ$.

7. $\tan \angle KLM = 3.0$. Hence, $\angle KLM \approx 72^\circ$.

8. $\tan \angle KLM = 1.5$. Hence, $\angle KLM \approx 56^\circ$.
This is more than half of the previous angle.

9. $\frac{200}{x} = \tan 59^\circ$ or $\frac{x}{200} = \cot 59^\circ$. Thus,

$x = \frac{200}{1.6643}$ or $x = 200(0.6009) \approx 120$.

10. (a) (1) $\cot \angle POQ = \tan \angle PQO = \frac{OP}{PQ}$.

(2) $\tan \angle POQ = \frac{PQ}{OP}$.

(3) $\frac{1}{\tan \angle POQ} = \frac{1}{\frac{PQ}{OP}} = 1 \div \frac{PQ}{OP} = 1 \cdot \frac{OP}{PQ} = \frac{OP}{PQ}$.

(4) By (1) and (3), $\cot \angle POQ = \frac{1}{\tan \angle POQ}$.

(b) No. $\sin \angle POQ = \frac{PQ}{OQ}$ and $\cos \angle POQ = \frac{OP}{OQ}$.

These are not reciprocals of each other.

11. Let x stand for the distance in miles to the lighthouse at 3 p.m. Then $\tan 52^\circ = \frac{x}{30}$. $x = 30 \tan 52^\circ \approx 30(1.28) \approx 38.4$.

Let r be the distance in miles between the ship and the lighthouse at 5 p.m. Then $\frac{30}{r} = \cos 52^\circ$. Hence,
$r = \frac{30}{\cos 52^\circ} \approx \frac{30}{.616} \approx 48.7$

(Other trigonometric ratios could be used. Some pupils may find the measurement of the other angle, 38°, and use trigonometric ratios of this angle. The answers, however, should be about the same.)

12. (a) 60°. Because all are congruent and the sum of their measures is 180.

(b)	$\overline{OB} \cong \overline{BC}$	Sides of an equilateral triangle are congruent.
	$\angle COB \cong \angle BCO$	Angles of an equilateral triangle are congruent.
	$\overline{OA} \cong \overline{AC}$	The mid-point of a segment separates the segment into two congruent segments.
	$\Delta OAB \cong \Delta CAB$	S.A.S.
(c)	$\angle OAB \cong \angle CAB$	They are corresponding angles of congruent triangles.
	$m\angle OAB = 90^\circ$	it is congruent to an angle, and the sum of their measures is 180.
(d)	$m\angle OBA = 30^\circ$	$180 - (90 + 60) = 30.$

(e) $(AB)^2 = \sqrt{(OB)^2 - (OA)^2} = \sqrt{2^2 - 1^2} = \sqrt{3} \approx 1.7321.$

*13. (a) $\frac{1}{2}$

(b) $y^2 = 3$

$y = \sqrt{3}$

(c) $\sin 60^\circ = \frac{\sqrt{3}}{2} \approx \frac{1.7321}{2} \approx 0.8660$

$\tan 60^\circ = \frac{\sqrt{3}}{1} \approx 1.7321$

*14. (a) $(DF)^2 = (DE)^2 + (EF)^2$

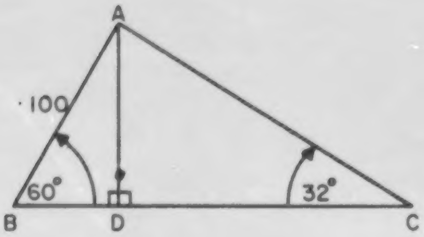
$= 1^2 + 1^2$

$= 2$

$DF = \sqrt{2}$

(b) $\sin 45^\circ = \frac{\sqrt{2}}{2} \approx \frac{1.4142}{2} \approx 0.7071$

*15.



In right triangle ABD	In right triangle ADC
$\frac{AD}{AB} = \sin 60^\circ$	$\frac{AD}{DC} = \tan 32^\circ$
$AD = 100 \sin 60^\circ$	$\frac{DC}{AD} = \cot 32^\circ$
$AD = 100(0.8660) = 86.6$ $= 87$	$DC = (86.6)(1.6003)$
to the nearest integer.	$DC = 138$ to the nearest integer.

9-4. The Slope of a Line

This section is included because of its close connection with the tangent of an angle, with earlier work on graphing, and with the later material on variation.

There is no attempt to find the slope of any lines except those through the origin, but in view of the more general definition of slope, it is important to connect the slope with the tangent of the angle rather than the ratio $\frac{y}{x}$. In general, the slope of a line is defined to be the tangent of the angle which the line makes with the X-axis. For instance, the line $y = x + 3$ makes an angle of 45° with the X-axis; hence, its slope is $\tan 45^\circ = 1$. The values of the ratios $\frac{y}{x}$ vary along this line as may be seen from the following table:

y	4	5	6	7	8
x	1	2	3	4	5
$\frac{y}{x}$	4	$\frac{5}{2}$	2	$\frac{7}{4}$	$\frac{8}{5}$

The slope of the line $y = mx + b$ is m, the coefficient of x.

Because these more general considerations are best dealt with later, we consider only slopes of lines through the origin but students should not be taught things here which they will have to unlearn later.

Later the connection between slope and direct variation will be exploited. The mention of "grade" is only incidental.

Answers to Class Exercises 9-4

1. (a) $\tan \angle AOB = \frac{3}{4}$ (b) $y = \frac{3}{4}x$

2. (a) $y = \frac{4}{1}x$ or $y = 4x$ (c) $\tan \angle COR = \frac{1}{4}$
   (b) $y = \frac{1}{4}x$ (d) $\tan \angle BOR = 4$

3. (a) $\overleftrightarrow{OD}$ (e) $\overleftrightarrow{OA}$
   (b) $\overleftrightarrow{OA}$ (f) $\overleftrightarrow{OD}$
   (c) $\overleftrightarrow{OC}$ (g) $\overleftrightarrow{OB}$
   (d) $\overleftrightarrow{OB}$ (h) $\overleftrightarrow{OC}$

4. (a) $\frac{1}{2}$ (c) 3
   (b) $\frac{7}{4}$ (d) 1

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Answers to Exercises 9-4

1. (a) $y = \frac{1}{4}x$

   (b) $y = \frac{1}{3}x$

   (c) $y = x$ or $y = (1)x$

   (d) $y = 2x$

   (e) $y = 5x$

   (f) $y = \frac{3}{5}x$

   (g) $y = \frac{7}{5}x$

   (h) $y = \frac{1}{3}x$

   (i) $y = \frac{0}{5}x$ or $y = 0$.
   (The line described here is the X-axis.)

2. (a) $\frac{1}{2}$

   (b) About 27°

3. (a) $\frac{7}{12}$

   (b) About 30°

4. (a) About 0.05

   (b) About 0.05

   (c) About 3°

   (d) $\frac{5}{100}$ or $\frac{1}{20}$

5. (a) 0.0349

   (b) 0.0349 (according to table. Actually the $\tan 2^\circ$ is a little more than $\sin 2^\circ$.)

   (c) About 3.5 feet

   (d) About 3.5 %

6. (a) About 64°

   (b) About 1000 feet

9-5. Similar Triangles

Our concern in this chapter is to give a rather precise definition of similar triangles, without the restriction to right triangles. It is easy for teacher and student (and text author!) to use the terms "corresponding angles" and "ratios of corresponding pairs of sides" rather loosely. Occasional use and reference to the precise definition given in this section is quite desirable. Students may find the topic of similar triangles quite difficult just because the words and terms we often use are used too carelessly.

Again, of course, there is no intent to provide proofs, or even informal arguments, for the properties stated in this section. The experience provided in Section 9-1 should make the conclusions appear reasonable. That is the best we can hope for in this kind of treatment. The logical points involved in proving these properties are among the most difficult and subtle of all elementary geometry and are left for the formal geometry course, usually studied in Grade 10.

Class Exercises 9-5 involve additional computations like that in the text to assist the pupil in understanding the definition of similar triangles. The corresponding angles of the figures are congruent as before, but the ratio of the corresponding sides varies for each pair. If the teacher feels the need of more examples along this line they can be supplied by extending the figure in the first part of this section. In fact, with the figure given, the triangles COD and EDF may also be compared.

Answers to Class Exercises 9-5

1. (a) $m(\angle A') = 30$, $m(\angle B') = 75$

   (b) $A'C' = 8$

   (c) $\frac{2}{3} = \frac{5}{B'C'}$, $B'C' = \frac{3}{2}(5) = 7\frac{1}{2}$. $A'C'$ cannot be determined since there is insufficient information given.

   (d) $A'B'$ cannot be determined--insufficient information. $\frac{4}{5} = \frac{B'C'}{3}$; $B'C' = \frac{4}{5}(3) = 2\frac{2}{5}$.

   (e) $m(\angle A') = 30$, $m(\angle C')$ cannot be determined--insufficient information.

2.

T (a) Alternate definition 1.

F (b) The third pair of corresponding sides are not necessarily congruent.

T (c) Alternate definition 2.

F (d) Insufficient information to make the claim that $\Delta ABC \sim \Delta A'B'C'$.

T (e) Alternate definition 2.

3. (a) No; for example, take a square and a rectangle which is not a square; the corresponding angles are equal, but the corresponding sides are not proportional.

(b) No; consider two parallelograms whose sides are proportional; their corresponding angles need not be congruent.

4. (a) When $A'B' = 6$, $A'C' = 12$ and $B'C' = 14$

(b) When $A'C' = 2$, $A'B' = 1$ and $B'C' = \frac{7}{3}$

(c) When $B'C' = 5$, $A'B' = \frac{15}{7}$ and $A'C' = \frac{30}{7}$

5. Yes. The triangles will be similar because the angles of one triangle are congruent to the corresponding angles of the other triangle.

6. Yes. If two angles of a triangle are congruent to corresponding angles of a second triangle, the third angle of the first triangle will correspond to and be congruent to the third angle of the second triangle. (If the measures of $\angle 1$ and $\angle 2$ are known, the measure of $\angle 3$ must be $180 - \left(m(\angle 1) + m(\angle 2)\right)$.)

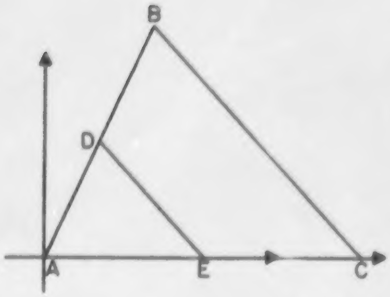
Answers to Exercises 9-5

1. (a) $\frac{AB}{AC} = \frac{AD}{AE}$. Corresponding sides of $\sim$ $\triangle$s are proportional.

   (b) $\frac{AB}{AD} = \frac{AC}{AE}$. Same reason.

   (c) $\frac{AD}{DE} = \frac{AB}{BC}$. Same reason.

   (d) $\frac{AD}{AC} \neq \frac{AB}{AE}$. They are not corresponding sides of similar Δs and are therefore not proportional.



2. No, the same conditions would hold, namely, corresponding sides of similar triangles are proportional.

3. (a) $\frac{AB}{AC} = \frac{AD}{AE}$

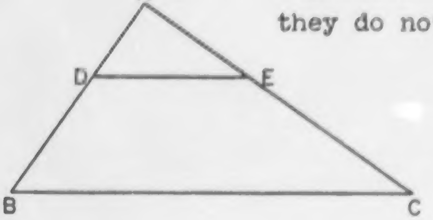
   (b) $\frac{AB}{AD} = \frac{AC}{AE}$

   (c) $\frac{AD}{DE} = \frac{AB}{BC}$

   If two Δs are $\sim$, the ratio of pairs of corresponding sides are equal.

   (d) $\frac{AD}{AC} \neq \frac{AB}{AE}$

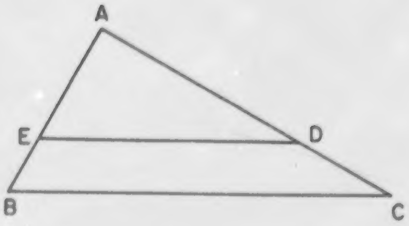
   They are not pairs of corresponding sides of $\sim$ Δs, therefore they do not form equal ratios.



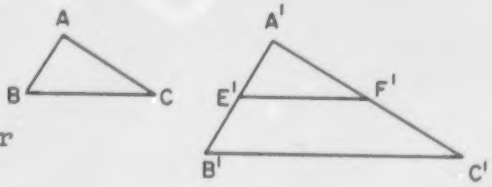
4. 90 - 31 = 59

5. 180 - (35 + 47) = 180 - (82) = 98

6. $\angle AED \cong \angle ABC$ and $\angle ADE \cong \angle ACD$. Since they are corresponding angles of parallel lines with $\overline{AB}$ and $\overline{AC}$ as transversals. $\angle BAC \cong \angle EAD$ since it is the same angle. Since the three angles of $\triangle AED$ are congruent to the corresponding angles of $\triangle ABC$, the two triangles are similar. $\angle AED$ corresponds to $\angle ABC$.



7. $\triangle ABC \cong \triangle A'B'C'$ (S.A.S.). Using the arguement from Problem 6, $\triangle A'E'F'$ is similar to $\triangle A'B'C'$. Therefore $\triangle ABC$ is similar to $\triangle A'B'C'$.



8. This can be shown in the same way as in Problem 7, except that the point E' is now located so that $A'B' = 3(A'E')$ and F' is located so that $A'C' = 3(A'F')$.

9. (a) Since $\frac{AB}{AC} = \frac{A'B'}{A'C'}$

and $A'B' = 3(AB)$,

$$\frac{AB}{AC} = \frac{3(AB)}{A'C'}.$$

Thus, $A'C' = 3(AC)$. (Note: the multiplication property of equality is used several times in going from the second to the third step.

Similarly, $\frac{AB}{BC} = \frac{A'B'}{B'C'}$

replacing A'B' with 3(AB),

$$\frac{AB}{BC} = \frac{3(AB)}{B'C'}.$$

Thus, $B'C' = 3(BC)$.

(b) The same method may be used to show that

$$A'C' = s(AC)$$

and $B'C' = s(BC)$.

9-6. Scale Drawings and Maps

This section is to be thought of as an application of similar triangles, not an exhaustive treatise on the construction of scale drawings and maps. This is the reason for the brief treatment.

The fundamental ideas here should already be familiar to the student but their sharpening is made possible by the theory that came before. A teacher may prefer to give a little more work on the reading of maps but it should not be carried much further lest it disrupt the continuity of the chapter and give too much prominence to problems far distant from similar figures.

Perhaps special comment should be made about the notation

1 inch = 1 mile.

This is really a correspondence--in no sense is it equality. But this is the usual way of expressing this correspondence and students should become familiar with it. There should be no confusion since obviously one inch is not the same as one mile.

Answers to Class Exercises 9-6

1. (a) About 17 cm.

   (b) About 340 feet

(c) $17^2 = 15^2 + 8^2$ or $289 = 225 + 64$

$340^2 = 300^2 + 160^2$ or $115{,}600 = 90{,}000 + 25{,}600$.

2. (a) About 18.8 inches

(b) 10.0 inches

3. (a) 25 inches (d) 5 inches

(b) 50 mm. (e) $2\frac{1}{2}$ inches

(c) $6\frac{1}{4}$ inches (f) 2.5 cm.

4. (a) 100 feet (d) 80 feet

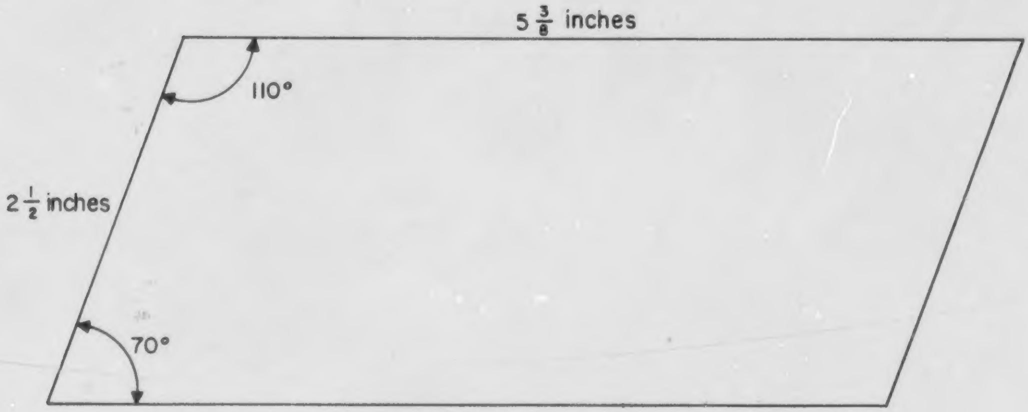
(b) 20 feet (e) 5 feet

(c) 80 feet (f) 200 feet

Answers to Exercises 9-6

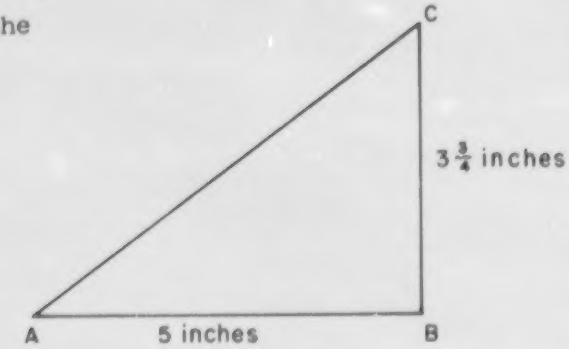
1. 850 miles 2. $3\frac{3}{4}$ inches

3.


$5\frac{3}{8}$ inches
110°
$2\frac{1}{2}$ inches
70°

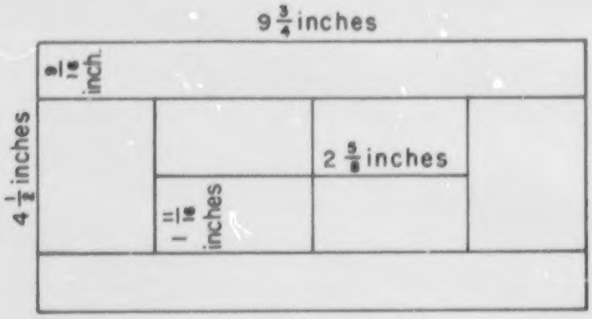
(Measurements are not as indicated on drawing.)

4. (Not to scale given in the text.)


C
3 3/4 inches
A
5 inches
B

The measurement of $\overline{AC}$ is $6\frac{1}{4}$ inches which represents 50 miles.

5.


9 3/4 inches
9/16 inch
4 1/2 inches
2 5/8 inches
1 11/16 inches

(Not to scale given in the text.)

6. $\frac{1,200,000}{12 \cdot 5280} \approx 18.9$ miles.

7. (a) About 66.4 feet

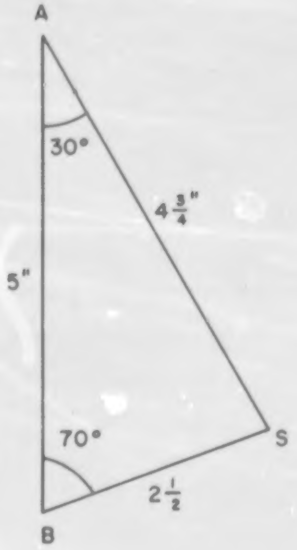
(b) He passes the pitcher's box about 3 feet away.

(c) About 105 feet

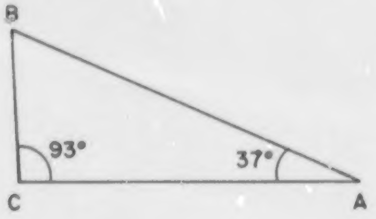
(d) About 42.8 feet

8. (a) The corresponding angles of the two triangles are congruent.

(b) The approximate measurements on the scale drawing are as indicated in the figure. Since 5 inches corresponds to 100 feet, the distances desired are approximately: BS = 50 feet, AS = 95 feet. Here some review of Chapter 5 is in order. Since one cannot measure inches to any greater precision than 0.1, the relative error in the answer could be as much approximately as 1 foot. Hence, there is no point in calculating it to more decimal places.



9.



(Not drawn to scale as required by problem.)

(a) AB = 5 inches
AC = $3\frac{3}{4}$ inches
BC = 3 inches

(b) $m(\angle BAC) \approx 37$

(c) City B is northwest of City A. (The compass direction is about 323°, but angles greater than 180° have not been discussed.)

10. (a) AB = 10 inches
AC = $7\frac{1}{2}$ inches
BC = 6 inches

(b) and (c) are same as for Problem 9.

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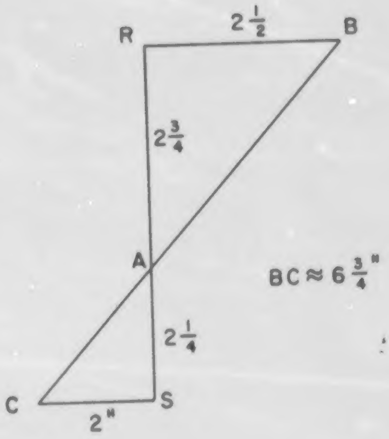
11. Here the measurements would be approximately those of the figure if on the map 5 inches corresponds to 100 feet.

(a) $BC \approx 135$ feet.

(b) $SA \approx 45$ feet.
$AR \approx 55$ feet.

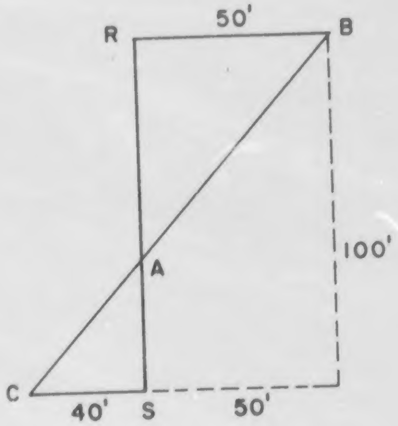

R
$2\frac{1}{2}$
B
$2\frac{3}{4}$
A
$BC \approx 6\frac{3}{4}$"
$2\frac{1}{4}$
C
S
2"

*12. Without a scale drawing, one may use the figure given and notice that BC is the hypotenuse of a right triangle with sides 100 and $40 + 50 = 90$, respectively. Hence, the length of the hypotenuse is $\sqrt{90^2 + 100^2} = 10\sqrt{181} \approx 135$. If we let x denote the distance SA, use similar triangles to get

$$\frac{50}{100 - x} = \frac{40}{x}, \quad 50x = 4000 - 40x,$$

$$90x = 4000, \quad x = \frac{400}{9} = 44\frac{4}{9}.$$

Similarly, for AR, or it may be noticed that $AR = 100 - x$.


R
50'
B
A
100'
C
40'
S
50'

9-7. Kinds of Variation

The concept of variation can be very appropriately introduced by use of tables of data and graphs as well as in connection with similar triangles. An attempt has been made in this section to use examples from science. It is hoped that one of the contributions of these two books for use in Grades 7 and 8 can be the encouragement of the practice of using applications of the mathematics to science as well as to the more traditional business and social topics.

Here several types of variation are used so that some differences among them can be observed. In the sections to follow specific attention will be given to direct and inverse variation and in an optional section to still other topics. Attention is called to the graph of $y = ax$ to show the relation of the second part of the chapter beginning with this section to the first part of the chapter on similar right triangles.

Only very special kinds of variation are considered in this and the following sections. The statement "y varies as x varies" could include any table of values of x and y in which not all the values of y are equal. The statement "y increases as x increases" could be made about any of the following relationships:

$y = x + 3$, $y = x^2$ with x positive, $y = \sqrt{x}$ with x positive.

But actually the term "y varies directly as x" or even "y varies as x" is defined to mean a very special kind of relationship; namely, when x is multiplied by any number, the corresponding value of y is multiplied by the same number. The same idea is expressed by writing that "x and y are proportional." By this time in this chapter, students should be familiar with this idea from the discussion of similar triangles.

Though the terminology is not used, we are really here considering some special and simple kinds of functions. Hence, not only are these sections important in themselves and for their connection with the previous part of this chapter but as an introduction to the idea of function which becomes so important in later mathematics.

Devotees of Winnie the Pooh may remember the line:

> "The more it snows, tiddlety pom,
> The more it goes, tiddlety pom,
> On snowing."

This might be interpreted to mean a much more complex relationship than any we have considered here; the rate of snowing is proportional to the amount it has already snowed. This is what is called "increasing exponentially" and example of which is continuous interest, a type of interest which is now beginning to appear in banking practice.

Answers to Class Exercises 9-7

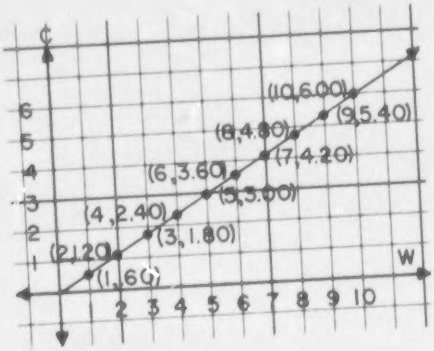
1.

w	Amount in pounds	0	1	2	3	4	5	6	7	8	9	10
c	Cost in dollars	$ 0	$ 0.60	$ 1.20	$ 1.80	$ 2.40	$ 3.00	$ 3.60	$ 4.20	$ 4.80	$ 5.40	$ 6.00

(a) cost increases

(b) 0.6 or 0.60

(c) c = 0.6 w

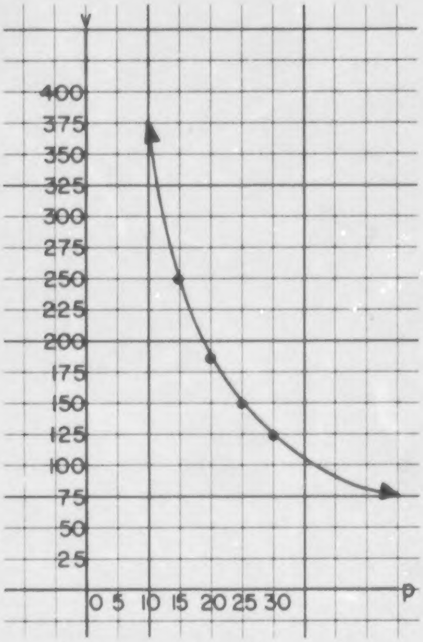


2. (a) If $P = 40$, $v \approx 94$.

   (b) The volume decreases.

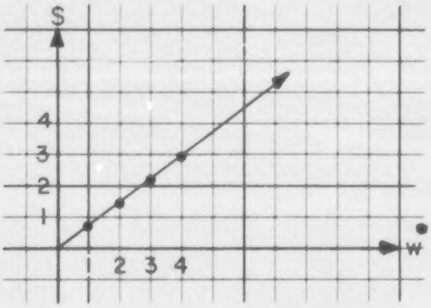
   (c) If $P = 10$, $v \approx 370$.

   (d) The pressure decreases.



Answers to Exercises 9-7

1.

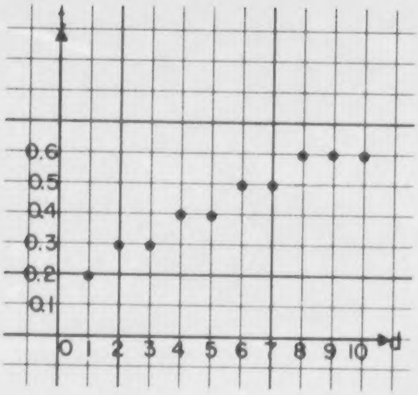


(a) S increases

(b) w increases

(c) When $w = 8$, $S = 6$.

2.



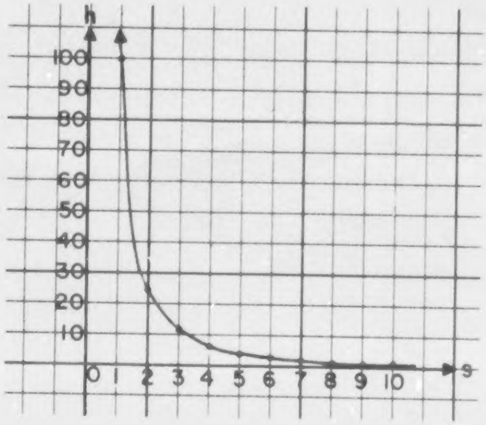
(a) Yes. As d increases, t tends to increase. Due to difficulty in measuring the time, the graph does not give a true picture of the actual relationship.

(b) The prediction can safely be stated as more than 0.6 seconds.

3.

s	1	2	3	4	5	6	7	8	9	10
h	$\frac{100}{1}$	$\frac{100}{4}$	$\frac{100}{9}$	$\frac{100}{16}$	$\frac{100}{25}$	$\frac{100}{36}$	$\frac{100}{49}$	$\frac{100}{64}$	$\frac{100}{81}$	$\frac{100}{100}$
	100	25	11.1	6.2	4	2.8	2.0	1.6	1.2	1

(a)



(b) $h = \frac{100}{s^2}$,

$s \neq 0$

[pages 395-396]

9-8. Direct Variation

One of the ways to talk about direct variation is through reference to the graph of a straight line. As x increases, y increases. This is true when negative values are considered for x as well as positive values. The increase in y for an increase of 1 in x is constant. This constant can be called the constant of proportionality. The constant of proportionality is also the ratio of the change in y, between two points on the graph of a straight line or between two of the elements in a set of data, to the change in x. If a value of y is multiplied by k the value of the corresponding x of an ordered pair (x,y) is also multiplied by k. The relation of direct variation can be expressed by the number sentence $y = ax$ for $a \neq 0$. Except for the number pair $(0,0)$ the ratio of number pairs, $\frac{y}{x}$, which satisfies the equation, is a. Since for number pairs in which the relation is direct variation, $\frac{y}{x} = a$, the set of numbers y and the set of numbers x are said to be proportional. Also, a is the slope of the line $y = ax$.

Properties like these enumerated in the above paragraph can be discussed carefully and in some detail for direct variation. Then in considering inverse and other kinds of variation it may be helpful to test these properties to see if they hold in the new situation.

Answers to Questions in Section 9-8

Multiply, the answer is $9.00.

$c = (0.60)(w)$. Multiplication.

Slope is 0.60. Now $(0.60)(10) = 6.00$ which is the increase in cost.

$\frac{c}{w}$ is the change in cost for each unit change in weight.

$(50)(1) = 50$: 50 miles in one hour.

$(50)(2) = 100$: 100 miles in two hours.

$(50)(3.5) = 175$: 175 miles in three and one-half hours.

$(50)(t) = 50t$: $50t$ miles in t hours.

[pages 396-397]

Answers to Class Exercises 9-8

1. (a) $k = \frac{3}{2}$

   (b)

w	0	1	2	3	4	5
s	0	$\frac{3}{2}$	3	$\frac{9}{2}$	6	$\frac{15}{2}$

   (c) yes

2. (a) $t = 32n$

   (b) $d = 0.32n$ or $d = \frac{32n}{100}$

   (c) yes

   (d) yes

   (e) 32 (for $t = 32n$)

Answers to Exercises 9-8

1. (a) $t = 33.9n$

   (b) $d = 0.339n$ or $d = \frac{33.9n}{100}$

2. (a) $n(2)$ or $2n$ feet

   (b) $d = 2n$

   (c) no, d will increase, not decrease.

3. (a) $i = 12f$ (b) i increases

4. For 1, $k_2 = 33.9$

   For 2, $k_3 = 2$

   For 3, $k_4 = 12$

5. Yes. $n \neq 0$; that is, n cannot be zero.

[pages 398-399]

6. $y = kx$
$6 = (k)(2)$
$3 = k$

7. $y = kx$
$^{-}3 = (k)(^{-}12)$
$\frac{1}{4} = k$

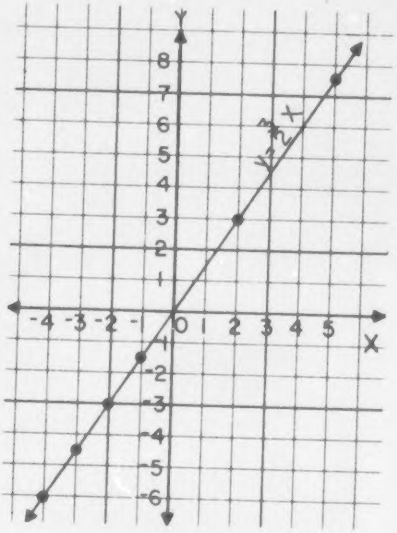
8. (a) Yes, when b increases, a increases, when b decreases, a decreases.

(b) $a = 10b$

9. $d = kt$
$240 = (k)(6)$
$40 = k$
$\therefore\ d = 40t$

10. (a) $(^{-}4, ^{-}6)$, $(^{-}3, ^{-}4\frac{1}{2})$, $(^{-}2, ^{-}3)$, $(^{-}1, ^{-}1\frac{1}{2})$, $(0, 0)$, $(2, 3)$, $(5, 7\frac{1}{2})$.

(b)



11. (a) Yes, y is doubled.
    (b) y is halved.
    (c) x is multiplied by 10.
    (d) Yes.

12. (a) x is halved. (b) y is tripled.

13. (a) Sample: $f = 5280\ m$, where m is the number of miles and f is the number of feet.
    (b) $M = (\frac{1}{100})(c)$, where c is the number of centimeters and M is the number of meters.

9-9. Inverse Variation

In the study of any concept in mathematics it is desirable to observe situations in which the properties associated with the concept do not hold as well as to study situations in which the properties hold. For this reason, if no other, it is highly desirable to introduce inverse as well as direct variation. In addition there are many important relations in science and in social situations which are illustrations of inverse variation. A number of these are included in the exercises.

It has been said that the arithmetic of Grades 1 through 4 is concerned with the relation $x + y = a$, while the arithmetic of Grades 5 through 8 is concerned with the relation $xy = a$. This statement is based on the important place in upper grade mathematics of such topics as multiplication and division, area, per cent problems, and rate problems. Interesting interpretations of inverse variation may be made in any of these situations.

This relation of inverse variation can be expressed by the number sentence $xy = a$, $a \neq 0$, or by $y = \frac{a}{x}$. In either case no value can be obtained for y if x is 0, or for x if y is 0. This introduces a new situation with graphs. The graph

[pages 400-402]

of xy = a does not cross the Y-axis or the X-axis. Hence, the graph is in two parts and is said to be discontinuous. It may be well to point out that y can be determined for x, any positive number, no matter how small, even for $x = 10^{-1000}$. Similarly, a value can be determined for y if x is any negative number, no matter how close the number is to zero. In higher mathematics the students will learn that the lines x = 0 and y = 0 are asymptotes of the hyperbola xy = a and that the hyperbola is said to approach the axes asymptotically.

Here again, only a very special kind of relationship is considered. We could say "y increases as x decreases" about any of the following relationships:

$$x + y = 3, \quad yx^2 = 1, \quad y = (\tfrac{1}{2})^x$$

assuming in the last two that x is a positive real number. But here we mean by "y varies inversely as x" that the proportion of increase of y is equal to the proportion of decrease of x; that is, if y is multiplied by a number, the corresponding x is divided by that same number.

It would probably be confusing to a class to note that "y varies inversely as x" means the same as "y varies directly as $\frac{1}{x}$" but the teacher should convince himself of this fact.

Answers to Exercises 9-9

1. (a)

r(number of miles per hr.)	10	20	25	50	60	75	80	100
t(number of hours)	10	5	4	2	$1\frac{2}{3}$	$1\frac{1}{3}$	$1\frac{1}{4}$	1

(b) (r)(t) = 100

(c) The time is halved.

(d) r decreases, such that the product of r and t is still 100.

2. (a)

s	10	12	15	16	30	40
n	24	20	16	15	8	6

(b) $(n)(s) = 240$ where both n and s are counting numbers only.

3. $WD = 36$

(a)

W	2	9	18	36	6	3
D	18	4	2	1	6	12

(b) If D is doubled, W is halved.
As W increases, D decreases, such that the product of W and D is still 36.

4. $(r)(p) = 200$
If the number p is doubled, the number r is halved.
If the number r is doubled, the number p is halved.

5. $k_1 = 100$
$k_2 = 240$
$k_3 = 36$
$k_4 = 200$

6. For direct variation the ratio of the variables is constant. For inverse variation, the product of the variables is a constant.

7. $xy = k$
$(2)(6) = 12$
$k = 12$

8. $xy = k$
$(\frac{1}{2})(10) = k$
$\therefore \quad k = 5$

9. No. The product of a and b is not a constant. The ratio $\frac{a}{b}$ is a constant. The table suggests that a and b are related by the equation $a = \frac{1}{2}b$.

10. (a) No.

    (b) No. They are related by $y = 2x^2$.

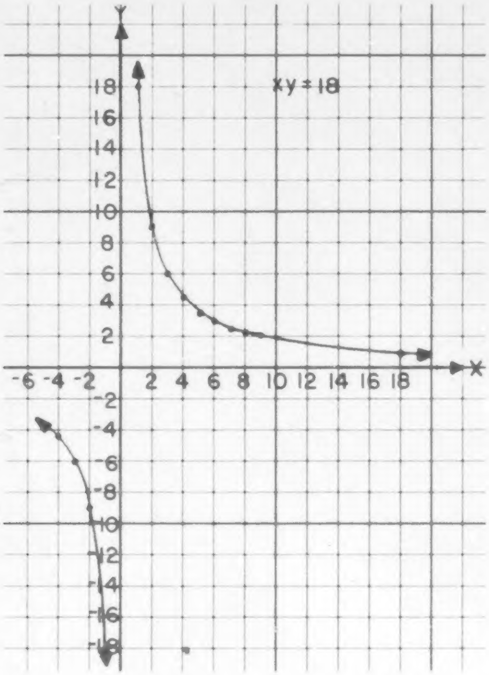
11. (a)

x	$^-4$	$^-3$	$^-2$	$^-1$	1	2	3	4	5	6	7	8	9	18
y	$^-4\frac{1}{2}$	$^-6$	$^-9$	$^-18$	18	9	6	$4\frac{1}{2}$	$3\frac{3}{5}$	3	$2\frac{4}{7}$	$2\frac{1}{4}$	2	1

    (b) No. If either factor is zero, the product would be zero. Since the product cannot be zero, neither factor can be zero.

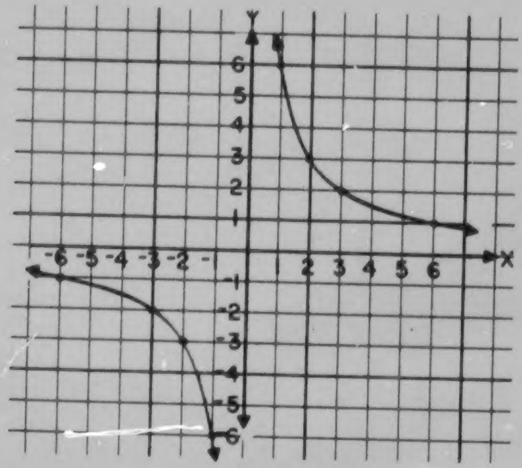
    (c) Yes.

    (d) Inverse.



12. $xy = 6$

x	$^{-}6$	$^{-}3$	$^{-}2$	$^{-}1$	1	2	3	6
y	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}6$	6	3	2	1



9-10. Other Types of Variation (Optional)

In better classes particularly, the teachers may wish to give at least brief attention to other types of variation. For this reason variation of one number in an ordered pair as the square of the other number and, analogously, variation of one of a pair, "inversely as the square" of the other are introduced. Also reference is made to the graph of $y = \sin x$ for $x = 10, 20, \ldots, 80$. This last topic might be considered in order to refer again to some of the material earlier in the chapter. Unfortunately, students do not have at this time the necessary background for the study of the interesting and exceedingly important properties of the graph of $y = \sin x$.

This section might be assigned for self study by the students who have made the most rapid progress with the earlier parts of the chapter, or to those students who show the greatest interest in future scientific study.

Answers to Questions in Section 9-10

If the number t is doubled, the number d is multiplied by 4. If t is tripled, d is multiplied by 9.

$k = 16$

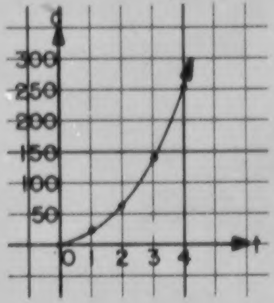
$d = 16t^2$

$d = 16(10)^2$

$d = 1600$

$\therefore$ the body falls 1600 ft. in 10 seconds.

Galileo: Born 1564 - Died 1642.



Answers to Exercises 9-10a

1. (a) 4 sq. in., 6 faces, 24 sq. in.

(b)

S	1	2	3	4	5	6
A	6	24	54	96	150	216

2. (a) $S = e^2$

(b) S varies as the square of e.

(c)

e	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
S	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

(d) (1) If $e = 3$,
$S = 9$

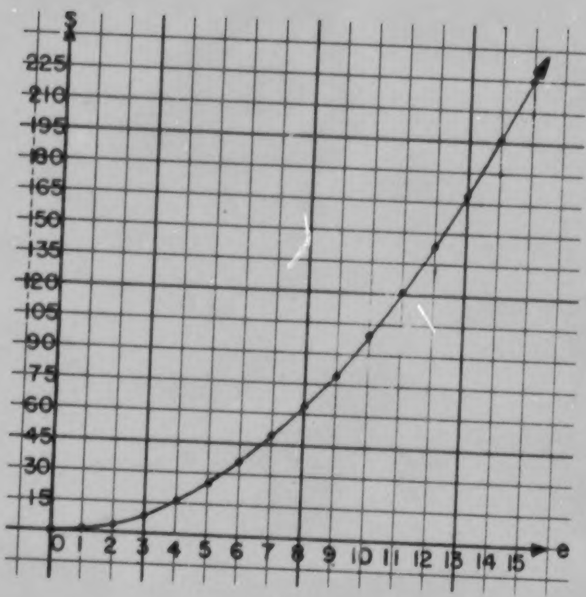
(2) If $S = 64$,
$e = 8$

(3) If $e = 5.5$,
$S \approx 30$

(4) If $S = 40$,
$e \approx 6.4$

(e) (1) If $e = 3$,
$S = 9$

(2) If $e = 5.5$,
$S = 30.25$



3.

d	1	2	3	4	5	6	7	8	9	10
t	0.2	0.3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.6
$\frac{d}{t^2}$	25	22.2	33.3	25	31.2	24	28	22.2	25	27.7

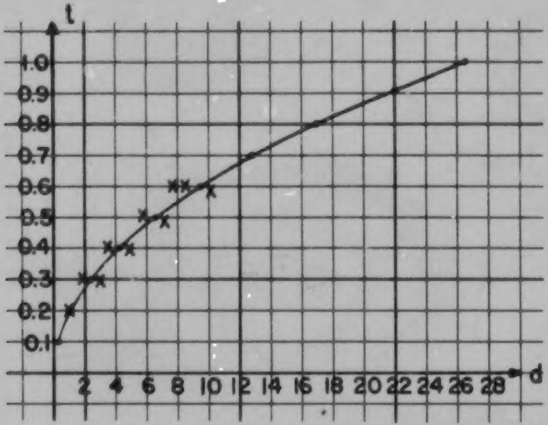
The mean of the ratios $\frac{d}{t^2}$ is 26.4

$d = 26.4t^2$

d	0.3	1.0	2.4	4.2	6.6	9.5	12.9	16.9	21.4	26.4
t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

Yes, the theoretical curve fits.

Yes, $\frac{d}{t^2}$ seems to remain constant according to her data.



x marks the observations
. marks points described by
$d = 26.4t^2$.

4. $E = kV^2$

$64 = k(4)^2$

$4 = k$

(a) $E = 4V^2$

(b) $E = 4(6)^2 = 144$

(c) $16 = 4V^2$

$4 = V^2$

$V = 2$

5. (a) $\frac{100}{280} = \frac{x}{1}$, $x = \frac{5}{14}$. $\therefore$ $\frac{5}{14}$ lbs. is required.

(b) $\frac{5}{14} \times (70) = 25$, $\therefore$ 25¢ is required.

(c) $C = KS^2$, $K = \frac{70}{280} = \frac{1}{4}$; $C = \frac{1}{4}S^2$

(d) $C = \frac{1}{4}(65)^2$

$C = 1056\frac{1}{4}$ ¢ or cost = \$10.57.

(e) $C = \frac{1}{4}s^2$

$C = \frac{1}{4}(75)^2 = 1406\frac{1}{4}$, Cost = \$14.07. Enough can be bought to sow the plot.

6. (a) $d = kt^2$

(b) $d = 9k$

(c) $144 = 9k$
$16 = k$
$d = 16t^2$

(d) $d = 16(5)^2$
$d = 400$ $\therefore$ the distance is 400 ft.

Answers to Exercises 9-10b

1. $P = 4e$
$S = 6e^2$
$V = e^3$

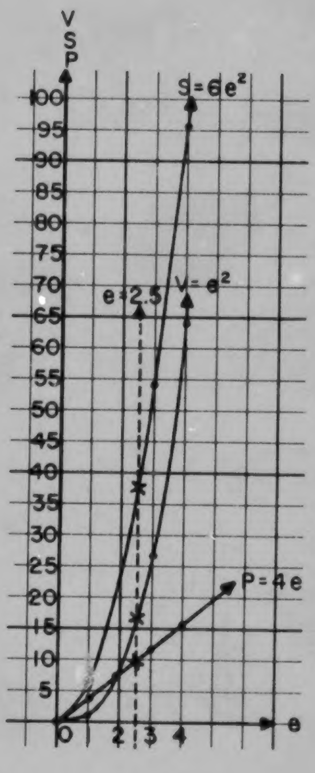
2. (1) P varies directly as e.

(2) S varies as the square of e.

(3) V varies as the cube of e.

3.

e	0	1	2	3	4	
P	0	4	8	12	16	for $P = 4e$
S	0	6	24	54	96	for $S = 6e^2$
V	0	1	8	27	64	for $V = e^3$



4. If $(e) = 2\frac{1}{2}$, $P = 10$
$S = 37\frac{1}{2} \approx 38$
$V = 15.625 \approx 16$

5. If $e = 10$, $P = 40$
$S = 6(10)^2 = 600$
$V = (10)^3 = 1000$

$\therefore$ if $e = 10$, $P < S < V$

9-11. Summary

The committee that prepared this chapter has no recommendation on the use of the summary. It is included for the convenience of teacher and students. The review exercises could be used as test questions, additional assignments, or general class summary discussion.

Answers to Exercises 9-11

1.
$y = kx$
$16 = k(2)$
$\therefore k = 8$ and $y = 2x$
$y = 2x$
$y = 2(5) = 10$

2.
$xy = k$
$(2)(16) = k$
$k = 32$
$\therefore xy = 32$
$(5)(y) = 32$
$y = 6\frac{2}{5}$

3.
$y = kx^2$
$16 = k(2)^2$
$k = 4$ and $y = 4x^2$
$y = 4(5)^2 = 100$

4. $\frac{A_1}{A_2} = \frac{d_1^{\,2}}{d_2^{\,2}} = \frac{(4)^2}{(6)^2} = \frac{16}{36} = \frac{4}{9}$

5. (a) $d = kp$

$5 = k(10)$ solving for k and d,

$k = \frac{1}{2}$ and $d = \frac{1}{2}p$

$14 = \frac{1}{2}p$ replacing d and k with 14 and $\frac{1}{2}$.

$\therefore p = 28$, pull must be 28 lbs.

(b) $d = \frac{1}{2}(14) = 7$, distance will be 7 in.

6. $\frac{1}{2}$ pint $= \frac{1}{16}$ gal.
$\therefore$ 7 lbs. per sq. in. for 1 gal. (7)(16) lbs. per sq. in. for $\frac{1}{2}$ pint.

or

$PV = K$

$(7)(1) = K = 7$. If $V = \frac{1}{16}$ then $P(\frac{1}{16}) = 7$.

$P = (16)(7) = 112$, hence, 112 lbs. per sq. in. is the pressure exerted by a certain amount of hydrogen in a 1 gal. jug if it is later enclosed in a half-pint jar.

7. In $x + y = k$, y does not vary inversely as x because the product of x and y is not constant.

8. Inverse variation, $xy = k$

9. Direct variation $y = kx$
π is the constant of proportionality.

10. $V = \pi r^2 h$
If r is multiplied by 5 and h is unchanged

$V = \pi(5r)^2 h$

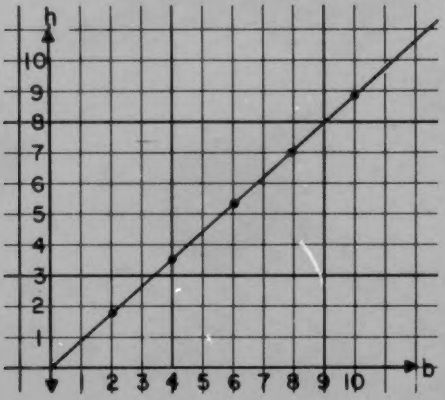
$\therefore V = 25(\pi r^2 h)$

hence, the volume is multiplied by 5.
25π is the constant

11. Using the relation $h = \frac{b}{2}\sqrt{3}$ we have:

b	2	4	6	8	10
h	$\sqrt{3}$	$2\sqrt{3}$	$3\sqrt{3}$	$4\sqrt{3}$	$5\sqrt{3}$
h $\approx$	1.7	3.5	5.2	6.9	8.7

The graph is a half-line.

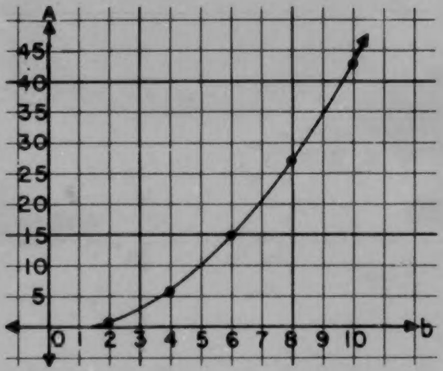


12. $h = \frac{b}{2}\sqrt{3}, \quad b > 0$

13.

b	2	4	6	8	10
A	$\sqrt{3}$	$4\sqrt{3}$	$9\sqrt{3}$	$16\sqrt{3}$	$25\sqrt{3}$
A $\approx$	1.7	6.9	15.6	27.7	43.3

The graph is the right half of the parabola $A = \frac{b^2\sqrt{3}}{4}, \quad b > 0.$



[page 416]

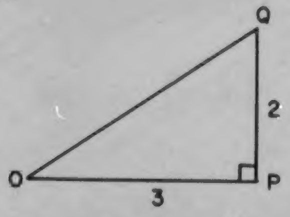
Sample Questions

The following questions are not intended to provide a single test for Chapter 9. They are intended only to help the teacher in preparing his own tests.

True-False

T 1. Triangle ABC is similar to triangle A'B'C' if $\frac{AB}{A'B'} = \frac{BC}{B'C'}$ and $m(\angle ABC) = m(\angle A'B'C')$.

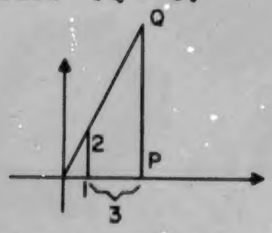
F 2. In the figure



$\sin \angle POQ = \frac{2}{3}$

F 3. $\text{Sin } 40^\circ = \frac{1}{\cos 40^\circ}$.

F 4. In the figure below $PQ = 6$.



F 5. The point (4, 5) lies on the line whose equation is $y = \frac{4}{3}x$.

T 6. The point (4, 5) lies on the line joining (0, 0) and (12, 15).

F 7. The tangent of an angle is always a number between 0 and 1.

T 8. The height of a tree may be calculated if the length of its shadow and the angle of elevation of the sun are known.

F 9. Any two right triangles are similar.

T 10. If $F = 14a$, F varies directly as a.

F 11. If $F = \frac{k}{6}$, F varies inversely as 6.

T 12. If x varies inversely with y, and y is 4 when x is 9, then $y = 2$ when $x = 18$.

F 13. If y varies directly as x then x varies inversely with y.

T 14. If the measures of two angles of one triangle are equal, respectively, to the measures of two angles of another triangle, then the third angles also have equal measure.

F 15. If two sides of one triangle have lengths equal, respectively, to the lengths of two sides of another triangle, then the third sides must also have equal lengths.

F 16. The sine of an angle varies directly with the measure of the angle.

Multiple Choice

Write the letter of the best answer.

C 1. If one acute angle of a right triangle has the same measure as an acute angle of another right triangle, then the triangles are

A. congruent

B. isosceles

C. similar

D. all of the above

E. cannot tell from this information

D 2. The lengths of the sides of a triangle are 6, 8, and 9 inches. Which of the following are the lengths of the sides of a triangle similar to it?

A. 3, 4, 6 inches
B. 2, 3, 4 inches
C. 9, 12, 16 inches
D. 18, 24, 27 inches
E. none of the above

C 3. The equation of the line through (0, 0) and (4, 5) is

A. $x = \frac{5}{4}y$
B. $y = \frac{4}{5}x$
C. $y = \frac{5}{4}x$
D. $x = \frac{4}{5}y$
E. none of the above

C 4. The slope of the line whose equation is $y = x$ is

A. 0
B. x
C. 1
D. it does not have any
E. none of the above

A 5. The slope of the X-axis is

A. 0
B. 1
C. $y = 0$
D. it does not have any
E. none of the above

D 6. The slope of the Y-axis is

A. 0
B. 1
C. $x = 0$
D. it does not have any
E. none of the above

C 7. The sine of an angle of a right triangle

A. depends on the length of one side only

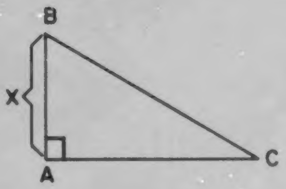
B. depends on the length of the hypotenuse only

C. depends on the measure of the angle only

D. all of the above

E. none of the above

D 8. In the drawing we can find the number x if we know



A. AC and BC

B. AC and m(∠ ACB)

C. AC and m(∠ ABC)

D. all of the above are correct

E. none of the above is correct

B 9. If you spend two dollars buying beans, the cost per pound of the beans and the number of pounds purchased provide an example of

A. direct variation

B. inverse variation

C. variation as the square

D. inverse square variation

E. none of the above

B 10. If p varies inversely with t and $p = 6$ when $t = 2$ then what is p when $t = 10$?

A. 30

B. $\frac{6}{5}$

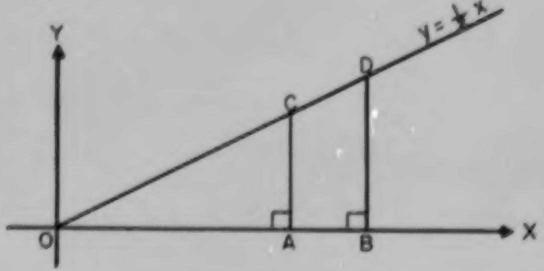
C. $\frac{6}{10}$

D. $\frac{5}{6}$

E. not enough information

Completion

For Problems 11-16 use this figure:



$\frac{5}{2}$ 11. The coordinates of point C are (5, ____).

8 12. The coordinates of point D are (____, 4).

∠ ODB 13. m(∠ OCA) = m(∠ ____)

$\frac{1}{2}$ 14. tan ∠ AOC = ________.

$\frac{1}{\sqrt{5}}$ 15. sin ∠ AOC = ________.

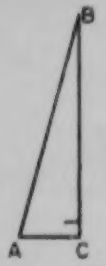
$\frac{1}{\sqrt{5}}$ 16. cos ∠ ACO = ________.

90 45 45 17. The measure of each of the three angles of an isosceles, right triangles are_____, ______ and _____.

75 18. A right triangle has a 15° angle. The measure of the other acute angle is _______.

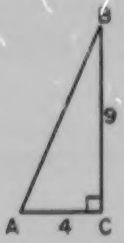
2 19. The slope of the line joining the points (2, 1) and (4, 5) is _____.

152 20. In the figure



tan ∠ A = 3.8 and $\overline{AC}$ has length 40 feet.
BC = _____.

For Problems 21-25 use this figure:



$\frac{9}{4}$ 21. tan ∠ A = _____.

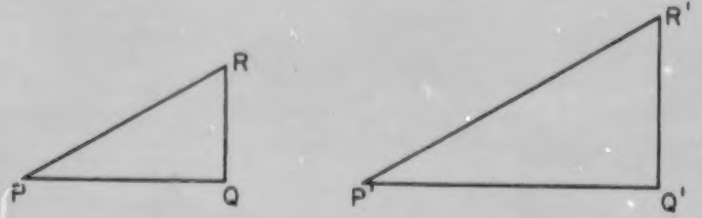
$\frac{9}{\sqrt{97}}$ 22. sin ∠ A = _____.

$\frac{4}{9}$ 23. tan ∠ B = _____.

$\frac{9}{\sqrt{97}}$ 24. cos ∠ B = _____.

$\frac{4}{9}$ 25. cot ∠ A = _____.

For Problems 26 and 27 use this figure:



$m(\angle Q) = m(\angle Q') = 90$ and $m(\angle P) = m(\angle P')$.

16 26. If $QR = 12$, $Q'R' = 15$, and $P'Q' = 20$, then $PQ =$ ______.

30 27. If $QR = 18$, $Q'R' = 27$, and $PR = 20$, then $P'R' =$ ______.

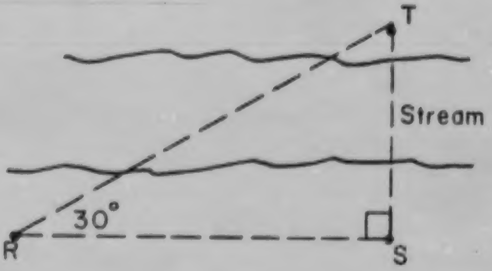
57.73 28. In the figure

Given:

$\sin 30^\circ = .5000$

$\tan 30^\circ \approx \frac{1.7321}{3} = .5773.$

$\cos 30^\circ \approx \frac{1.7321}{2} = .8660$



S and T represent trees on opposite sides of the stream. The distance between R and S is 100 feet. The distance between S and T is ____ feet.

115.46 29. In Problem 18, $RT =$ ______.

inversely 30. If $y = \frac{\pi}{x}$, then y varies _______ with x.

Chapter 10

NON-METRIC GEOMETRY

Many teachers have found that when Chapter 10 is taught before Chapter 11 that Chapter 11 is much easier for the pupils, in fact that Chapter 10 is so helpful that several days teaching time on Chapter 11 can be saved. After a very careful tryout of this chapter over a two-year period, the authors of the text strongly recommend that Chapter 10 be included as a regular part of the course.

The material of this chapter will be new to almost all pupils. There are a number of reasons for including it in eighth grade mathematics. Among these are:

1. It helps develop spatial intuition and understanding.
2. It emphasizes in another context the role of mathematics in reducing things to their simplest elements.
3. It affords other ways of looking at objects in the world about us and raises fundamental questions about them.
4. It illustrates types of mathematical (geometric) reasoning and approaches to problems.
5. It gives an interesting insight into the meaning of dimension.

The general purposes of this chapter are similar to those for Chapter 4 of Volume I. It is suggested that teachers read the beginning of the Commentary for Teachers of that chapter.

In this chapter there is more emphasis on the use of models in understanding spatial geometry. There is correspondingly less emphasis on "sets" as such. This chapter is more geometry than "sets." However, the terminology of sets, intersections, and unions is used throughout this chapter. The students should already have some familiarty with these ideas. A quick run-over of the ideas of Chapter 4 of Volume I would be useful as background for most students. However, if sets, intersections, unions, and some geometrical terms can be explained as you go along, this material should be teachable without much specific background on the part of students.

The use of terms like "simplex" and the distinctions between 2- and 3-dimensional polyhedrons should contribute to precision of thought and language.

Time. The material of this unit is recommended for study for a period of about two weeks. If class interest is high, some extra time could profitably be used.

Reading. The text is written with the intention that the pupils read (and study) it. There may be occasional passages which some pupils will not follow easily. Pupils should be encouraged to read beyond these passages when they occur, and then go back and study them. Teachers may find it profitable to read some of the material with the pupils.

Materials. In teaching subject matter like this to junior high school students, models are of considerable use. Familiarity and facility with models should increase teaching effectiveness as well as improve basic understandings. The students should be encouraged to use cardboard or oak tag for their models of tetrahedrons and cubes. They will be asked to draw on the surfaces of these later. It is suggested that each student contribute one model of a regular tetrahedron and one of a cube for later class use. These will be needed in the study of polyhedrons. Each student should keep a model of a cube and a model of a tetrahedron at home for use in homework. There are several suggestions for making models from blocks of wood. The boys in the class should be encouraged to make these.

[pages 417-418]

10-1. Introduction

10-2. Tetrahedrons

It is probably desirable for the teacher to follow the instructions in the text to make in class either a model of a regular tetrahedron or a model of a non-regular tetrahedron or both.

Have each student leave one model of a tetrahedron with you. Have them keep one model of a tetrahedron to use at home.

Some boys might like to make models of tetrahedrons by sawing wood blocks.

Answers to Questions

The edges of (PQRS) are (PQ), (PR), (PS), (QR), (QS), and (RS).

The faces of (PQRS) are (PQR), (PQS), (PRS), and (QRS).

Answers to Exercises 10-2

1. (Construction)
2. (Construction)
3. The measure of the new angle DAC must not be more than the sum of the measures of angles BAC and BAD nor less than their differences. Otherwise, it won't fit. (This could be tried in class.)

10-3. Simplexes

The plural of simplex is also correctly written as "simplices" (which is similar to vertex and vertices). We choose "simplexes" because the term simplex itself is a new word which the students should learn to use with ease.

In Chapter 4 we regarded points, lines, and planes as our "building blocks." Here (with some prior notion of lines and planes) we will use points, segments, triangular regions, and solid tetrahedrons as our "building blocks."

The discussion about "taking points between" and "dimension" probably will need to be read more than once by pupils. After one gets the general ideas, the material is quite readable. A class discussion of these concepts prior to the reading of the text by the pupils may be a desirable practice.

The discussion of "betweenness" illustrates an important fact about reading some mathematics. One sometimes has an easier time reading for general ideas and then rereading to fill in details.

Answers to Class Exercise 10-3

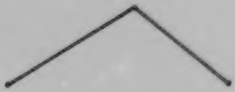
1. (a and b) The figures are given with this problem in the student text. In (a) the figure is a triangle. In (b) the figure represents the union of a triangle and its interior.
   (c) No; 2 (in drawing the sides of the triangle and in shading the interior of the triangle.)
2. (a) The union of the edges of the tetrahedron (ABCD).
   (b) The union of the faces (including the edges and vertices) of the tetrahedron (ABCD).
   (c) Solid tetrahedron.
   (d) No; 3 (in (a), (b), and (c).)
3. (a) The point itself
   (b) 0

Answers to Exercises 10-3

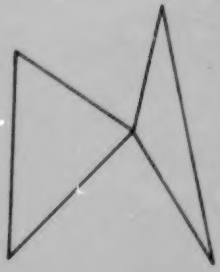
1. (a) 3 (b) 3

2. 2

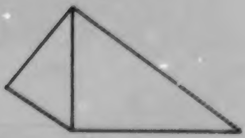
3.



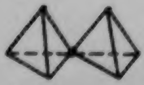
4.

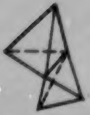


5.



6 and 7. The better students might be encouraged to draw figures for each of these problems, as:





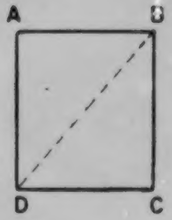
8. Place two tetrahedrons together so that the intersection is exactly one face of each. (See the third of the four figures in a row at the beginning of Section 10-5.)

10-4. Models of Cubes

Answers to Exercises 10-4

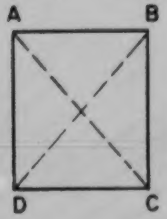
1. Constructions

2. There are a number of possible answers ... (PABD) and (PBCD) could be two of the 12 if one face is labeled as on the right.



3. 24 2-simplexes.

4. 24; Yes, you can make the comparison. The top vertices of the six pyramids would correspond to the six points in the middle of the faces of the cube.



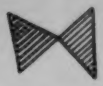
10-5. Polyhedrons

In class it would be a good idea to hold or fasten models of tetrahedrons together to indicate various 3-dimensional polyhedrons. Also, blocks of wood with some corners and edges sawed off (and maybe with a wedge-shaped piece removed) are good models of 3-dimensional polyhedrons.

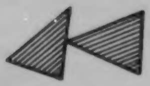
Models of tetrahedrons should be held together to indicate how the intersection of two could be exactly an edge of one but not an edge of the other. (They would not intersect "nicely.") Other examples can also be shown here.

Answers to Exercises 10-5

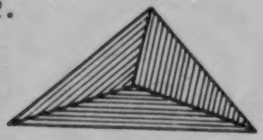
1. (a)



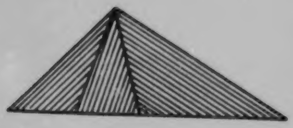
(b)



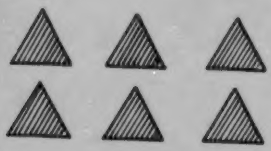
2.



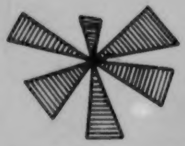
or



3. (a)

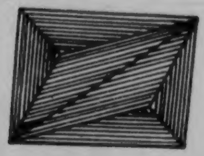


(b)



(one possible answer)

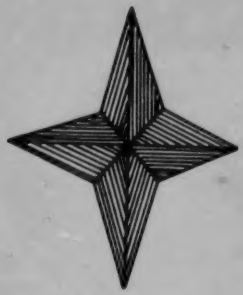
(c)



4. Draw the segments (BD), (GE), and (HE).

5. (a) Draw segments (BD), (BH), (DF), (FH), and either (BF) or (HD).

(b)



6. (a) (PQR), (XYZ), (XPQ), (XYQ), (YZQ), (ZQR), (XZP), and (PRZ).
   (b) (FPQR), (FXYZ), (FXPQ), (FXYQ), (FYZQ), (FZQR), (FXZP) and (FPRZ).
   (c) (QXYZ), (PXZQ) and (RPQZ)

10-6. One-Dimensional Polyhedrons

The students may enjoy drawing all sorts of simple closed polygons. Some should be in a plane. But the students also should be encouraged to draw some simple closed polygons on surfaces of cubes, tetrahedrons, wooden blocks, and so on.

It may be desirable to have a little practice in representing on a plane a few 1-dimensional polyhedrons which are not contained in a plane.

Answers to Questions

The polygonal paths from P to S are PS, PRS, PRQS, PQS, PQRS. Five in all.

Other simple closed polygons containing BE and GA are:

BEDFGAB
BEDCHFGAB
BEGAHCB
BEGAHFDCB
BEDFGAHCB

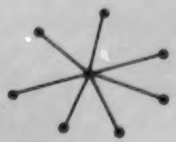
Answers to Exercises 10-6

1. 2 paths from A to B.
   3 simple closed polygons.

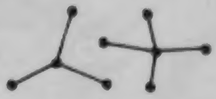
2. (a) 7
   (b) PQSP, PRSP, PQRP, QRSQ, PQRSP, PRQSP, PRSQP.
   (c) PQRSP (We are naming the vertices in order returning to a starting point.)

3. PABCDEFQ, PCBAFEDQ, PEDCBAFQ (there are others)

4.



or



or others.

5. There are many ways of doing it.

10-7. Two-Dimensional Polyhedrons

This section gives a good opportunity to draw many 2-dimensional simplexes and polyhedrons and to illustrate again sets of 2-simplexes which intersect nicely.

Some drawings and illustrations should be in the plane and some should be on surfaces or 3-dimensional polyhedrons.

Answers to Questions

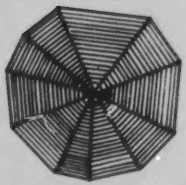
Students are asked if they see a relationship among the faces, edges, and vertices of certain polyhedrons. It would be difficult to discover Euler's formula from just three cases. The relationship will be developed in detail in Section 10-9.

Answers to Table and Exercises 10-7

1.

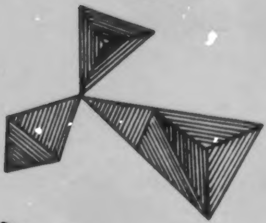
	F	E	V
Tetrahedron	4	6	4
Cube	6	12	8
Cube (with simplexes)	12	18	8

2. (a)



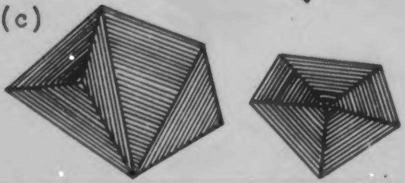
and many other possibilities.

(b)



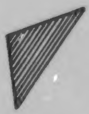
and many other possibilities.

(c)

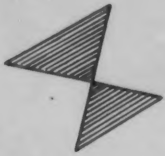


and many other possibilities.

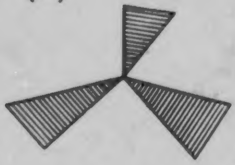
3. (a)



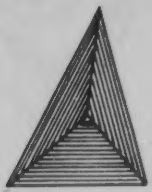
(b)



(c)



4.

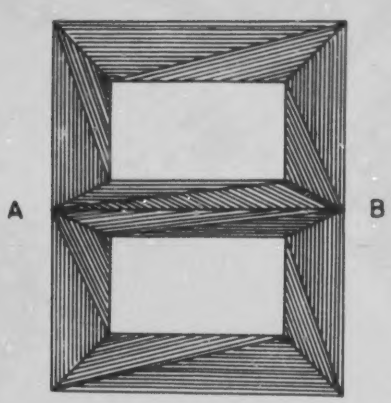


No, there won't be four. The union of the first three must itself be a 2-simplex. Yes, a tetrahedron.

5. $F = 24$ $E = 36$ $V = 14$.

6. 6 sets of 4-simplexes each.
12 original edges, and 6 sets of 4 new edges each.

7. This can be done in many ways. We have first subdivided it along AB into two figures like the second in the text of this section.



*8. (a) True

(b) False. This is true only of a 1-simplex whose endpoints are a vertex of the 2-simplex and a point of the opposite 1-simplex which is not a vertex of the 2-simplex.

(c) False. This is true only of a 2-simplex containing a 1-simplex (edge) of the 3-simplex and a point of the opposite edge which is not a vertex of the 2-simplex.

10-8. Three-Dimensional Polyhedrons

In this section and the next section there are many fine opportunities to use the models which the pupils have prepared. It is suggested that the class be divided into three or four groups and that each group take several models of regular 3 inch tetrahedra and fasten them together (with cellulose tape or paste) to produce some rather peculiar looking 3-dimensional polyhedrons. They can also fasten models of cubes together to produce odd looking polyhedrons.

Again there is a good opportunity here to encourage some of the boys in the class to make models of 3-dimensional polyhedrons out of blocks of wood. Start with a block of wood and saw corners or edges off of it and possibly notches out of it. The surface obtained will be a simple surface as long as no holes are punched

[pages 440-444]

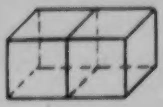
through the solid. Be sure that the surface is made up only of flat portions. A surface like this could be covered with paper and then colored or drawn on. It could then be re-covered with paper for further use. It is interesting that no matter how the block is cut up (without holes) every simple closed polygon on it will separate the surface into just two pieces. Also the models can be used for examples of Euler's formula in the next section. The faces, edges, and vertices can be counted rather easily on objects like this.

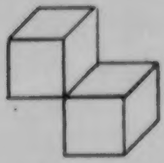
Make a model of a "square doughnut" out of 8 cubes as suggested in the text. The surface will not be a simple surface.

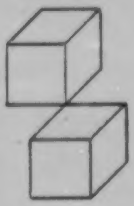
Answers to Exercises 10-8

1 and 2. Constructions.

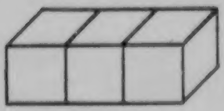
3. Three:

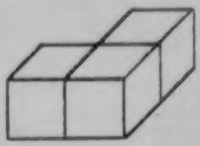






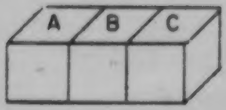
4. Two:



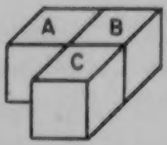


5. (a) Five of the many possibilities are as follows:

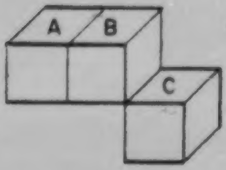
(1)



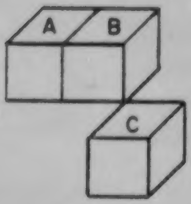
(2)



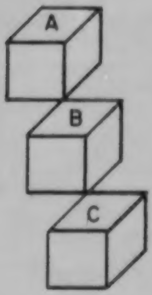
(3)



(4)



(5)



Intersections:

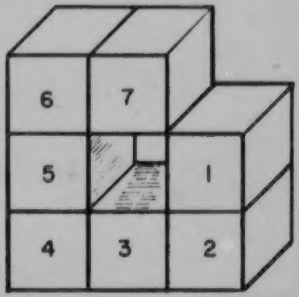
	A and B	A and C	B and C	A and B and C
(1)	face	∅	face	∅
(2)	face	edge	face	edge
(3)	face	∅	edge	∅
(4)	face	∅	vertex	∅
(5)	vertex	∅	vertex	∅

∅ represents the empty set.

(b) No.

(c) Yes.

*6.



The intersection of cube # 7 with cube # 6 is a face and the intersection of cube # 7 with cube # 1 is an edge. Thus, the figure is not a "simple surface."

10-9. Counting Vertices, Edges, and Faces--Euler's Formula

Use models of tetrahedrons and cubes in explaining the material of this section to the class.

The counting of faces, edges, and vertices on any simple surface should be interesting and informative for the pupils. Use models made of wood for some of the examples.

Use the model of the "square doughnut" of the last section to help pupils count faces on this type of surface.

The teacher or student should subdivide the surface of a model of a cube in an irregular way and count faces, vertices, and edges as indicated in the text.

In traditional solid geometry texts a polyhedron is usually defined as a solid formed by portions of plane surfaces, called faces of the polyhedron. It is assumed that the faces are enclosed by polygons. A theorem, credited to the German mathematician Euler, is usually included in the text. The theorem states that if the number of edges of a polyhedron is denoted by E, the number of vertices by V, and the number of faces by F, then $E + 2 = V + F$. In this section we see that the formula does not hold in general for surfaces which are not simple. The traditional treatment usually does not make a point of this limitation of the formula. Actually the "square doughnut" described in the text satisfies the traditional definition of polyhedrons to which reference was made.

In Chapter 13 it is noted that Euler's formula was known by Descartes who lived some one hundred years before Euler.

Regular Polyhedrons

Some of the best students will enjoy a study of why there are only five regular polyhedrons.

Recall that a regular polyhedron is defined to be a polyhedron all of whose faces are congruent regular polygons. One of the most surprising facts of solid geometry is that there are only five regular polyhedra. Since the usual proof of this fact uses polyhedral angles, the impression is often given that it is a metric property. But it is not such a property. This can be shown by giving a proof depending only on Euler's formula:

1. $V + F - E = 2$

and the following three other assumptions:

2. Each vertex is the end-point of the same number of edges.
3. Each face is bounded by the same number of edges.
4. Each face has at least three edges and each vertex is the end-point of at least three edges.

Here is the proof. Assuming Property 2, we may let r stand for the number of edges for each vertex. Then the product rV counts each edge twice, since each edge has two end-points. Hence,

(a) $$rV = 2E, \quad \text{or} \quad V = \frac{2E}{r}.$$

Assuming Property 3, we may let s stand for the number of edges which each face has. Then the product sF counts each edge twice, since each edge is on the boundary of two faces. Hence,

(b) $$sF = 2E, \quad \text{or} \quad F = \frac{2E}{s}.$$

Next, assuming Euler's formula, we replace V and F in this formula by the expressions in (a) and (b) and have

$$\frac{2E}{r} + \frac{2E}{s} - E = 2, \text{ that is,}$$

(c) $$E\left(\frac{2}{r} + \frac{2}{s} - 1\right) = 2.$$

Since E is positive,

$$\frac{2}{r} + \frac{2}{s} - 1 > 0,$$

that is,

$$\frac{2}{r} + \frac{2}{s} > 1, \text{ or}$$

(d) $$\frac{1}{r} + \frac{1}{s} > \frac{1}{2}.$$

If $\frac{1}{r}$ and $\frac{1}{s}$ were both less than or equal to $\frac{1}{4}$, Property (d) would be denied. Hence,

Either $\frac{1}{r}$ or $\frac{1}{s}$ is greater than $\frac{1}{4}$.

But $\frac{1}{r} > \frac{1}{4}$ means $4 > r$. Hence,

(e) Either $r < 4$ or $s < 4$ or both.

But Property 4, affirms that both r and s must be greater than or equal to 3. This, together with (e) shows:

(f) At least one of r and s is 3, and the other is 3 or greater.

Now, using (f) we can make a table listing the various possibilities.

r	3	3	3	3	4	5	6
s	3	4	5	6	3	3	3
$\frac{1}{r}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
$\frac{1}{s}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{r} + \frac{1}{s}$	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{8}{15}$	$\frac{1}{2}$*	$\frac{7}{2}$	$\frac{9}{15}$	$\frac{1}{2}$*

[pages 445-448]

The starred entries do not satisfy Condition (d). Furthermore, if r is 3 and s is greater than 6, the sum of the two reciprocals will also not satisfy Condition (d). This will also happen if s is 3 and r is greater than 6. Hence, there are just five possibilities:

1. $r = 3 = s$. Each vertex has three edges and each face has three sides and, hence, is an equilateral triangle. Then Formula (c) gives $E = 6$, (a) gives $V = 4$ and (b) gives $F = 4$.

$$E = 6, \quad V = 4, \quad F = 4.$$

The polyhedron is called a tetrahedron.

2. $r = 3$, $s = 4$. Each vertex has three edges and each face has four sides and, hence, is a square. Then Formulas (c), (a), (b) give

$$E = 12, \quad V = 8, \quad F = 6.$$

The polyhedron is a cube.

3. $r = 3$, $s = 5$. Each vertex has three edges and each face has five sides and, hence, is a pentagon. Then Formulas (c), (a), (b) give

$$E = 30, \quad V = 20, \quad F = 12.$$

This polyhedron is called a dodecahedron. ("dodeca" signifies "twelve")

4. $r = 4$, $s = 3$. Each vertex has four edges and each face has three sides and, hence, is an equilateral triangle. Then Formulas (c), (a), (b) give

$$E = 12, \quad V = 6, \quad F = 8.$$

This polyhedron is called an octahedron. ("octa" signifies "eight")

5. $r = 5$, $s = 3$. Each vertex has five edges and each face has three sides and, hence, is an equilateral triangle. Then Formulas (c), (a), (b) give

$$E = 30, \quad V = 12, \quad F = 20.$$

This polyhedron is called an icosahedron.

[pages 445-448]

We have merely shown that these five are the only possibilities. Models may be made to show that each of these polyhedrons do exist.

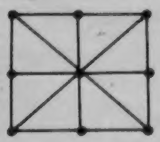
It is interesting to compare the second with the fourth, that is, the cube with the octahedron. The number of edges in each case is the same but the values of r and s, V and F are interchanged. This means that the number of faces of the octahedron is equal to the number of vertices of the cube and vice versa, while the number of edges is the same for both. These are sometimes called "dual" polyhedrons. The same connection exists between the dodecahedron and the icosahedron. The tetrahedron is its own dual.

There are also "semi-regular polyhedrons." Here Properties 1 and 4 still hold and the faces are regular polygons but not all the polygons are congruent. For instance, one such polyhedron has some faces which are squares and some which are equilateral triangles. It can be formed by "cutting off the corners" of a cube by planes through the mid-points of the edges. There will then be six squares on the faces of the cube and eight equilateral triangles at the corners. Each vertex will be the end-point of three edges. This is only one of many possible semi-regular polyhedrons.

Answers to Exercises 10-9

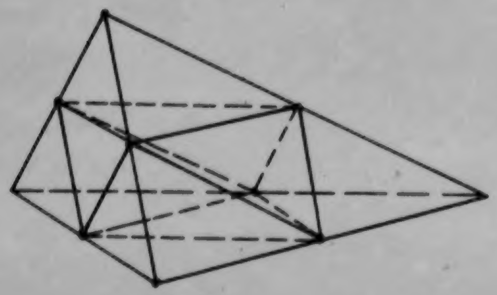
1. The count is 12 faces, 18 edges, and 8 vertices.
   The 2-simplexes of the subdivision of the tetrahedron can be corresponded to the faces of the solid which is the union of the five tetrahedron.

2. A face looks like
   There are:
   26 vertices,
   48 faces,
   72 edges (48 on interiors of faces of cube and 24 on edges of cube).



3. V + F - E should be 2.

4.



V = 10 V + F - E = 10 + 16 - 24 = 2.
F = 16
E = 24

5. (a) V = 16
F = 14
E = 28

V + F - E =
16 + 14 - 28 = 2

(b) V = 16
F = 14
E = 28

V + F - E =
16 + 14 - 28 = 2.

6. (a) V = 20
F = 18
E = 36

V + F - E = 2

(b) V = 18
F = 16
E = 32

V + F - E = 2.

Sample Questions

The following questions are not intended to provide a single test for Chapter 10. They are intended only to help the teacher in preparing his own tests.

True-False

T 1. A 2-simplex has three 1-simplexes as edges.

T 2. The surface of a cube can be regarded as the union of 2-simplexes.

T 3. If two simplexes of the same dimension intersect nicely, then the intersection must be a face, or an edge, or a vertex of each.

F 4. Every 1-dimensional polyhedron must be contained in a plane.

F 5. Every polyhedron is the union of a finite number of simplexes of the same dimension, each pair of which intersects nicely.

T 6. If a polyhedron is the union of simplexes which intersect any way at all, then the same set of points (the same polyhedron) is also the union of simplexes which intersect nicely.

F 7. The union of the edges of a 3-simplex (solid tetrahedron) contains exactly 4 simple closed polygons.

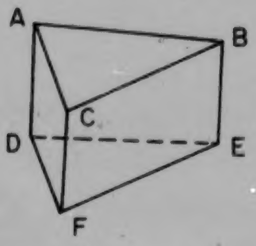
T 8. Any union of 3-simplexes, each pair of which intersects nicely, is a 3-dimensional polyhedron.

F 9. Euler's formula states that for any simple surface, $F + E - V$ equals 2.

T 10. The number of 1-simplexes (edges) on any simple closed polygon is equal to the number of vertices.

Other Sample Questions

(Use this triangular prism in answering Problems 1-3.)



1. List four polygonal paths from A to F.

2. List two simple closed polygons on the prism which do not lie in the same plane.

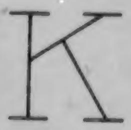
3. Which of the following are 2-dimensional polyhedra:
   (a) (ABC)
   (b) The union of (ABC) and (CBEF).
   (c) The union of all five faces of the solid.
   (d) The union of (ABC) and (DEF).
   (e) The union of (ACFD), (BCFE), and (ABED).

4. Do two tetrahedrons with exactly one vertex of each as the intersection determine a simple surface?

5. For a pyramid with a hexagonal base, find V + F - E, where V is the number of vertices, F the number of faces, and E the number of edges.

6. Think of a 3-dimensional polyhedron consisting of five tetrahedrons, in which the intersection of one tetrahedron with each one of the others, is a vertex. The other 4 have no intersections with each other.
   (a) How many vertices and edges does this polyhedron have?
   (b) Is the polyhedron a simple surface? Why?

7. Think of the letter K printed as shown.



(a) How many 1-simplexes are in this figure?

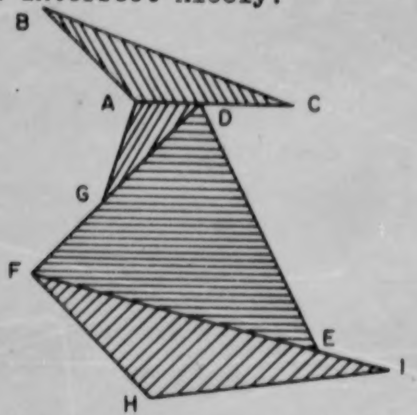
(b) Do the 1-simplexes intersect nicely?

(c) Is this figure a 1-dimensional polyhedron?

8. Sketch a 1-dimensional polygonal path consisting of three 1-simplexes which is not contained in a plane. Label the vertices so that you can state which part of the path is intended not to be in the plane of the paper.

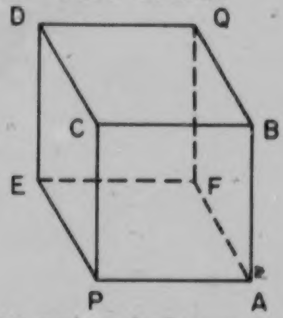
9. Why is a simple closed polygon necessarily the union of two polygonal paths? Draw figures to illustrate your statement.

10. The figure below represents a polyhedron as the union of 2-simplexes without nice intersections. Draw in three segments which will make the polyhedron the union of 2-simplexes which intersect nicely.

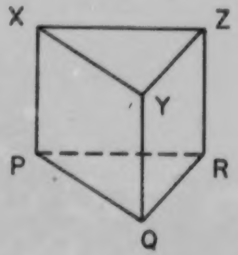


11. Draw four 2-simplexes whose intersection is one point such that (a) the point is a vertex of each, (b) the point is a vertex of three of them but not the fourth one.

12. List any five of the twenty-four 2-simplexes whose union is the surface of the cube below.

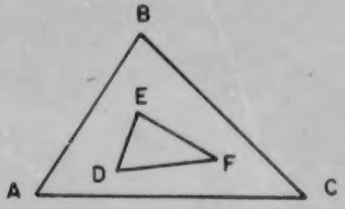


13. Using the triangular prism below name two polygonal paths from P to Z each of which contains all the vertices of the prism.

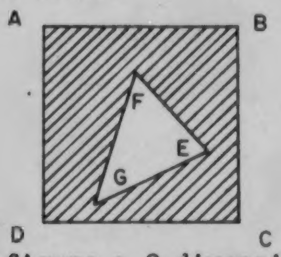


14. Think of the model of a pentagonal prism on which you have marked seven points, one at the center of each face. Consider each face to be subdivided into 2-simplexes each having the center point as a vertex. Determine the values of F (the number of 2-simplex), E (the number of 1-simplexes), and V (the number of 0-simplexes) for this subdivision of the whole surface, and compute V + F - E.

15. Show the polyhedron below as the union of six 2-simplexes which intersect nicely.



16. The figure below shows a square and its interior. In the interior a triangle has been marked. The interior of the triangle has been removed.



    (a) Is the shaded figure a 2-dimensional polyhedron? Why?

    (b) List all simple closed curves drawn in the figure.

    (c) List four polygonal paths drawn in the figure.

17. A 1-simplex is a set of 0-simplexes.

    (a) How can we think of a 2-simplex as a set of 1-simplexes?

    (b) How can we think of a 3-simplex as a set of 2-simplexes?

Answers to Other Sample Questions

1. The union of (AC) and (CF) is a polygonal path from A to F.
   The union of (AB), (BE), and (EF) is a polygonal path from A to F.
   The union of (AC), (CB), (BE), and (EF) is a polygonal path from A to F.
   These are several of many possibilities.

2. ABC and DEF, for example; or ACB and CBEF

3. All

4. No

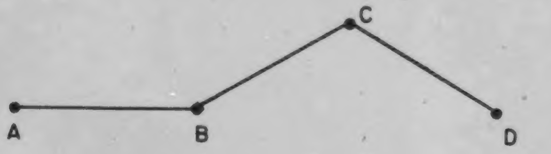
5. V = 7, F = 7, and E = 12, hence, 7 + 7 - 12 = 2.

6. (a) V = 16 E = 30

(b) This polyhedron is not a simple surface. In a simple surface the intersection of 3-simplexes must be a simple closed polygon.

7. If you think of the seven lines drawn by the draftsman, the answers are: (a) 7; (b) No.
If you think of the lines joining the 14 end-points and points of intersections, the answers are: (a) 13; (b) Yes.
In either case the answer to (c) is yes.

8.

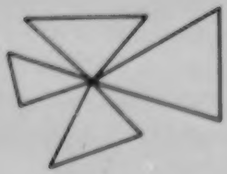


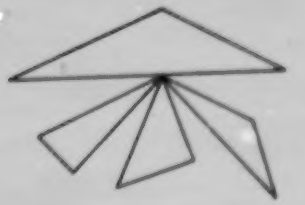
A and D are not in the plane of the paper and B is not in plane ACD.
This may be illustrated on a 3-simplex also, as in figure for Problem 1.

9. A simple closed polygon may always be thought of as the union of the two polygonal paths whose intersection is exactly their end-points.
The simple closed polygon ABCD is the union of (AB) and (BC) which is the polygonal path from A to C and (CD) and (DA) which is the polygonal path from C to A. The end-points A and C are the same for both polygonal paths.

10. Segments BD, GE, and HE.

11.



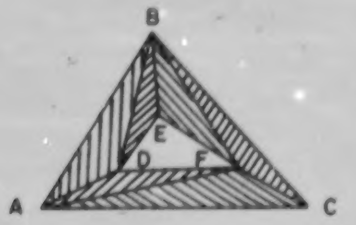


12. DCB, DBQ, DEC, CEP, PAC, for example.

13. PRQYXZ and PXYQRZ

14. V = 17, E = 45, F = 30. 17 + 30 - 45 = 2.

15.



16. (a) Yes. It is the union of seven 2-simplexes.
    (b) ABCD and FEG
    (c) AB, ABC, ADCB, etc.
        FE, FG, FEG, etc.

17. (a) A 2-simplex is the set of all 1-simplexes (segments) with end-points contained in the 2-simplex.
    (b) A 3-simplex is the set of all 2-simplexes with vertices contained in the 3-simplex.

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Chapter 11

VOLUMES AND SURFACE AREAS

General Remarks

The development of the main portions of this chapter is along empirical and intuitive lines. We have made generous provisions for discovery activities. The entire chapter is built around the construction and frequent use of the various models the pupils are to study.

At the end of this chapter you will find diagrams for constructing models of the geometric solids we will consider. We suggest you use the following steps to be taken in the construction of the models:

(a) Place a sheet of stiff paper such as oak tag behind the diagram in the text.

(b) With a compass point or pin point perforate the text page at all the vertices.

(c) Remove the stiff paper and join the holes by using a sharp pencil and a straightedge so that your diagram resembles that of the text.

(d) With a pair of scissors cut along the boundary lines of the figure or the stiff paper.

(e) Fold your cut-out along the inner edges of your solid.

(f) Use glue to paste the tabs on the inner edges of your solid.

(g) Seal each such edge by using tape, since some models are to be used as containers.

(h) Take good care of your models since they are to be used frequently.

We suggest that you take some time in class helping the pupils construct the first models. Be sure that they construct their models carefully. You will notice that in some cases a base is not sealed since such models will be used as containers. It is convenient at times to attach this base temporarily also.

We have assumed that your pupils are somewhat familiar with the following geometric terms: point, line, line segment, ray, parallel lines, intersecting lines, perpendicular lines, angles, measure of an angle, triangle, altitude and base of a triangle, base and altitude of a parallelogram, square, regular polygon, hexagon, perimeter of a polygon, circumference of a circle, Pythagorean property, congruent triangles, surface areas, and volumes.

You might perhaps take an inventory to see whether or not this is true. If preliminary work needs to be done we recommend Chapters 7, 8, 10, and 11 of the SMSG Mathematics for Junior High School, Volume I. The subject of measurement is extensively developed in Chapters 7 and 8. The subject of parallels, parallelograms, triangles, and right prisms is covered in Chapter 10, and circles and cylinders in Chapter 11.

If the pupils already know a good deal about areas of plane figures and volumes of right prisms respectively, Sections 11-1 and 11-3 could serve as a brief review of the main ideas.

If on the other hand your pupils have had little or no preparation for area and volume, Sections 11-1 and 11-3 should definitely be supplemented. In these sections we have done more reviewing than developing since we were not introducing the ideas of area and volume for the first time. Chapters 7, 8, 10, and 11 of SMSG Mathematics for Junior High School, Volume I, should most certainly be consulted in expanding upon the brief treatment in Sections 11-1 and 11-3 of this chapter.

Encourage your pupils to estimate areas and volumes prior to calculations and experiments. We feel that these activities will help them to appreciate more the ideas of area and volume than the results obtained by calculations alone. It will be necessary for you to supply such problems, as they have not been included in the text.

The activities of finding volumes by fitting inch cubes which pupils have constructed, and of pouring sand and salt into the interiors of the various prisms, should help them to understand the concept of volume.

You will notice that we have not done much articulating with the ideas developed in Chapter 10. We suggest you take advantage of these ideas whenever they will help you to clarify and unify the basic ideas presented in this chapter.

The dimensions were given for the various patterns for your convenience, but in the drawings they are merely proportional to the indicated measurements.

It is suggested that about 13 days be allowed for this chapter.

Answers to Exercises 11-1a

1. 48 sq. in.
2. $3\frac{3}{4}$ sq. ft.
3. 169 sq. in.
4. $12\frac{1}{4}$ sq. ft.
5. 240 sq. in.
6. 32.5 sq. cm.
7. 23.92 sq. ft.
8. 65.1 sq. cm.
9. 703 sq. ft.
10. Approx. 64 sq. in.
11. (a) 10 sq. in.
    (b) 70 sq. in.
12. 20 sq. in.

Answers to Exercises 11-1b

1. 76 sq. in.
2. 576 sq. in.
3. 247 sq. cm.
4. Approx. 61 sq. ft.
5. Approx. 12 sq. ft.

*6. 24 in.

*7. (a) 51,480 sq. ft. (b) $15,444.00

Answers to Exercises 11-1c

1. 1,038 sq. in.
2. 2,750 sq. in.
3. 17,400 sq. ft.
4. 3,080 sq. in.

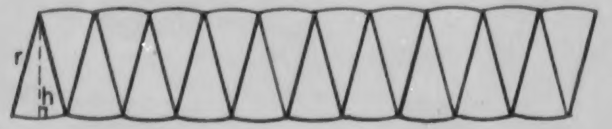
Answers to Exercises 11-1d

1. (a) 25π sq. in.
   (b) 100π sq. in.
   (c) 400π sq. in.
   (d) 20.25π sq. in.
   (e) 225π sq. in.
   (f) 196π sq. in.

2. 25π, 100π, 400π.
   Doubling the radius quadruples the area.

3. BRAINBUSTER.

   (a) Arrange the 20 congruent triangles as in the following figure



$$A_{▱} = bh$$

$$h \approx r; \quad b \approx \frac{1}{2}C = \frac{1}{2}(2\pi r) = \pi r$$

$$A_{\odot} \approx bh = (\pi r)(r) = \pi r^2$$

(b) We would say that the area of the circle is approximately equal to the area of the circumscribed polygon. It will always be less than the area of the circumscribed polygon since there will always be vertices of the polygon which are not points of the circle. Therefore, there is always some portion of the interior of the circumscribed polygon which is not contained in the interior of the circle. However, for larger values of n the areas are very close and we can think of the area of the interior of the circle as the lower limit of the area of interior of the circumscribed polygon.

Thus:

(Area of inscribed polygon) < (Area of circle) < (Area of circumscribed polygon). For very large values of n we can replace ($<$) by ($\approx$).

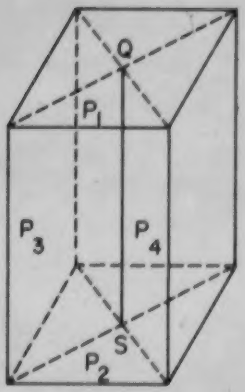
Answers to Exercises 11-2

1. Listing examples.
2. Examination of models.
3. Construction and examination of models.
4. ℓ_1 and ℓ_2 cannot be skew lines since they lie in the same plane r. ℓ_1 and ℓ_2 cannot intersect since they lie in parallel planes p_1 and p_2. Hence, ℓ_1 and ℓ_2 can only be parallel.
5. (a) A line perpendicular to a plane is perpendicular to all lines in that plane.
   (b) The side opposite the right angle of a right triangle is the hypotenuse of the right triangle.
   (c) The hypotenuse of a right triangle is the longest side of a right triangle.

6. (a) Draw P_3 and P_4 containing $\overline{QS}$.

   (b) P_3 and P_4 are perpendicular to P_2 since they contain $\overline{QS}$ a line perpendicular to P_2.

   (c) P_3 and P_4 are perpendicular to P_1 since P_1 is parallel to P_2.



   (d) $\overline{QS}$ is perpendicular to P_1 since P_3 and P_4 contain $\overline{QS}$.

11-3. Right Prisms

Some explanation should perhaps be given for the use of the capital letter, B, for the measure of the area of the base. The reason is that ℓ, w and h are in linear units, while B is in square units. There seems to be some virtue in using different kinds of letters to emphasize this distinction. However, this is not an important point and if the teacher prefers the small letter, there is no reason why this change should not be made.

Answers to Exercises 11-3

1. (a) 4
   (b) 10
   (c) $4\frac{1}{2}$

2. (a) 16 (c) $16\frac{1}{2}$
   (b) 28

3. Surface area is $33\frac{1}{2}$ square inches.
   Volume is 9 cubic inches.

4. Then ℓwh is replaced by $\ell w(2h)$ and, hence, the volume is doubled. Since the sum of the areas of the lateral faces is

$$2(\ell h + wh),$$

it is replaced by $2(\ell \cdot 2h + w \cdot 2h)$, which is twice the first area.

5. If ℓ and w are each doubled, the area of the base is multiplied by 4 and, hence, the volume is multiplied by 4. But, as above, we can see that the sum of the areas of the lateral faces is doubled.

6. If every dimension is doubled, the volume is multiplied by 8, the sum of the areas of the lateral faces by 4 and the surface area by 4.

7. Construction. Surface area is 29.8 square inches. Volume is 11.5 cubic inches.

8. (a) Perimeter of base of Model 4 is 6 inches.
   Perimeter of base of Model 5 is 6 inches.
   Perimeter of base of Model 7 is 6 inches.
   Perimeter of base of Model 8 is 6 inches.
   (b) Yes.
   (c) For rhombus based right prism, 7.8 cubic inches.
   For right rectangular prism, 9 cubic inches.
   For hexagon based right prism, 10.4 cubic inches.
   For right circular cylinder, 11.5 cubic inches.

(d) No

(e) See (c)

(f) The interior of a circle has the largest area.

9. Volume for Model 4, 9 cubic inches.
Volume for Model 6, 9 cubic inches.

(a) Yes. They are equal.
Perimeter of base of Model 4, 6 inches.
Perimeter of base of Model 6, $7\frac{1}{4}$ inches.

(b) No. The perimeters are not equal.

Answers to Exercises 11-4

1. Class activity.

2. Yes. It is the perpendicular distance between the bases.

3. No. It is not the perpendicular distance between the bases.

4. Too large, since the lateral edge of an oblique prism is greater than the height.

5. All the lateral edges of a prism must be congruent because each face is a parallelogram and, hence, has congruent lateral edges. Thus, one may move around the edges of the prism in order and see that each is congruent to its successor.

6. If two adjacent faces of a prism are rectangles, their common edge makes right angles with two edges of the base of the prism. Hence, their common edge is perpendicular to the base. Since all lateral edges are parallel in a prism, all lateral edges are perpendicular to the base. This shows that the prism is a right prism.

7. If a lateral edge of a prism is congruent to its altitude, it must be perpendicular to the base and, hence, from the previous problem, the prism is a right prism.

8. If the altitude of a prism is doubled and its base unaltered, Bh becomes $B(2h) = 2Bh$. Hence, the volume is doubled.

9. If all edges of a prism are doubled and its shape left unchanged, the area of the base is multiplied by 4 and the volume multiplied by 8.

Answers to Exercises 11-5

1. (a-c) Draw line segments $\overline{QS}$ and $\overline{QT}$. Consider triangles AQS and AQT to be right triangles since AQ is perpendicular to the base.
Now $\overline{AS}$ and $\overline{AT}$ are hypotenuses of their respective triangles. Therefore, $\overline{AS}$ and $\overline{AT}$ are 5 inches in length (3, 4, 5 right triangles).

   (d) Yes.

   (e) Yes.

2. (a) It would work for any regular polygonal region.

   (b) It would apply to any lengths.

   (c) Lateral edges.

3. (a) Use the Pythagorean property.

   (b) $(AS)^2 = (AQ)^2 + (SQ)^2$

   $(13)^2 = 12^2 + (SQ)^2$

   $169 - 144 = (SQ)^2$

   $25 = (SQ)^2$ same development for QT

   $5 = (SQ)$ $5 = QT$

(c) Yes (e) Yes

(d) Yes (f) Yes

4. (a) No

(b) Lateral edges, regular

5. Construction

6. (a) Four (b) Equal

7. 600 square inches

8. (a) 340 square inches (b) 13 inches

9. (a) 360 square feet *(b) $\sqrt{194}$

Answers to Exercises 11-6

1. (a) 28 cu. in. (c) 288,000 cu. ft.

(b) 800 cu. cm.

2. 3 cu. in. 3. 12 meters

4. 82,944,000 cu. ft.
3,072,000 cu. yd.

*5. $144\sqrt{3}$ sq. in.

*6. If the side is doubled and height halved, then the formula $V = \frac{1}{3} \times S^2 \times h$ becomes

$$\begin{aligned} V &= \frac{1}{3} \times (2s)^2 \times \frac{h}{2} \\ &= \frac{1}{3} \times 4s^2 \times \frac{h}{2} \\ &= \frac{2}{3} \times s^2 \times h \\ &= 2 \times \left(\frac{1}{3} \times s^2 \times h\right). \end{aligned}$$

Therefore the volume is doubled.

Answers to Exercises 11-7

1. $T = \pi r^2 + \pi rs$ or $T = \pi r(r + s)$

2. Lateral area is 36π sq. ft., total area is 45π sq. ft.

3. Radius is 9 feet, lateral area is 135π sq. ft., total area is 216π sq. ft., volume is 324π cu. ft.

4. Height is 36 inches, slant height is 39 inches, lateral area is 585π sq. inches.

5. Since the circumference is $2\pi r$, we have

$$\frac{1}{2}cs = \frac{1}{2}(2\pi r)s = \pi rs.$$

6. The area of each face will be $\frac{1}{2}bs$, where b is the length of the base. Then the lateral area of the pentagonal pyramid will be

$$5(\tfrac{1}{2}bs) = \tfrac{1}{2}(5b)s.$$

But $5b = p$ and, hence, our result follows.

7. If our pyramid had ten faces instead of 5, the number of faces would be 10 and 10b would be the perimeter of the base. This would hold for any number of sides. That is, suppose the base is a regular polygon of n sides, then its perimeter would be nb and the sum of the lateral areas would be

$$n(\tfrac{1}{2}bs) = \tfrac{1}{2}(nb)s = \tfrac{1}{2}ps.$$

8. The "perimeter" of a circle is its circumference and, hence, comparing the two formulas, we see that c corresponds to p. Thus, the two formulas correspond.

9. If the angle is 216°, the area of the sector of the circle will be $\frac{216}{360} = \frac{3}{5}$ of the area of the circle. Since the radius of the circle is 5, the area of the sector is

$$\frac{3}{5}\pi 5^2 = 15\pi.$$

This is also the lateral area of the cone, with 5 as its slant height. Then

$$L = \pi rs = 5\pi r = 15\pi.$$

This shows that $r = 3$ and, hence, the altitude of the cone is 4". Then the volume of the cone is $\frac{1}{3}\pi 3^2 \cdot 4 = 12\pi$.

10. Construction.

Sample Questions

True-False

T 1. A square is a parallelogram.

F 2. A trapezoid may have two pairs of parallel sides.

F 3. If the base and height of a triangle are both doubled, the area is multiplied by 2.

F 4. If the radius of a circle is doubled, the area is also doubled.

T 5. A regular decagon can be separated into ten congruent triangles.

F 6. The bases of a triangular right prism can be only right triangles.

T 7. Any two parallel faces of a rectangular right prism may be considered as the bases of the prism.

F 8. The length of the lateral edges of an oblique prism is the distance between its bases.

T 9. The volume of a pyramid varies directly as its height.

F 10. The volume of a cone is one-third that of a cylinder whose bases are congruent to the base of the cone.

T 11. A tetrahedron is always a pyramid.

F 12. A pyramid is always a tetrahedron.

F 13. A regular pyramid must be a tetrahedron.

T 14. Some pyramids have altitudes which do not intersect the closed region of the base.

T 15. The slant height of a right circular cone is never equal to the height of the cone.

Completion Questions

1. If adjacent sides of a rectangle are congruent, the figure is a square.

2. If two, and only two, opposite sides of a quadrilateral are parallel, the figure is a trapezoid.

3. If adjacent sides of a parallelogram are congruent, the figure is a rhombus or square.

4. If each side of a rectangle is doubled, its area is multiplied by 4.

5. If the height of a triangle is doubled, its area is multiplied by 2.

6. If an octagon is separated into congruent triangles by connecting the center with each vertex, the area is 8 times the area of each triangle.

7. The area of a polygon inscribed in a circle $<$ ($<$, $>$, or $=$) area of the circle.

8. If the radius of a circle is doubled, the area is multiplied by __4__.

9. If the radius of a circle is multiplied by 3, the area is multiplied by __9__.

10. If all sides of a polygon have sides of equal measure, and all angles have equal measure, it is called a __regular__ polygon.

11. In the blank following the name of each figure write the formula used to find its area.

    (a) Parallelogram __$A = bh$__

    (b) Rhombus __$A = bh$__

    (c) Triangle __$A = \frac{1}{2} bh$__

    (d) Circle __$A = \pi r^2$__

    (e) Trapezoid __$A = \frac{1}{2}h(b_1 + b_2)$__

    (f) Regular polygon __$A = \frac{1}{2}hp$__

12. The length of the base of a rhombus is 12 inches and the area is 6 square inches. The altitude is __$\frac{1}{2}$ in.__

13. A trapezoid has an altitude of 7 inches, and the average of the two bases is 14 inches. The area is __98__ square inches.

14. One of the five congruent triangles into which a regular pentagon is divided has an area of 13.5 square inches. The area of the pentagon is __67.5__ square inches.

15. If two planes are parallel the distances from different points of one plane to the other plane are __equal__.

16. If a line is perpendicular to one of two parallel planes, it is also __perpendicular__ to the second plane.

17. The measure of the edge of a cube is 4 inches. The surface area is __96__ square inches.

18. The formula for the volume of a cube is __$V = s^3$__.

19. A triangular right prism has for its bases right triangles the lengths of whose sides are 6 inches, 8 inches and 10 inches. The lateral edges measure 15 inches. The surface area is 408 square inches.

20. The volume of the prism in Problem 19 is 360 cubic inches.

21. An oblique prism has square regions for bases whose edges are 3 inches in length. The distance between the bases is 7 inches. The volume is 63 cubic inches.

22. The radius of the base of a right circular cylinder is 8 feet and its altitude is 35 feet. The surface area is 688π square feet.

23. In Problem 22, the volume is 2240π cubic feet.

24. In Problem 22, if the cylinder is open at one end the surface area is 624π square feet.

25. The following solids have bases that are equal in measure, and volumes that are equal in measure. How do their altitudes compare? (Use $<$, $>$, and $=$).

(a) A cylinder and a cone.

The altitude of the cylinder $<$ the altitude of the cone.

(b) A prism and a cone.

The altitude of the prism $<$ the altitude of the cone.

(c) A prism and a cylinder.

The altitude of the prism $=$ the altitude of the cylinder.

(d) A pyramid and a prism.

The altitude of the pyramid $>$ the altitude of the prism.

(e) A cone and a pyramid.
The altitude of the cone $\underline{\quad = \quad}$ the altitude of the pyramid.

(f) A cylinder and a pyramid.
The altitude of the cylinder $\underline{\quad < \quad}$ the altitude of the pyramid.

26. A cone whose base radius measures 60 cm. has a slant height which measures 1 meter. The lateral area is $\underline{\quad 6000\pi \quad}$ sq. cm.

27. A cone whose base radius is 2 inches has an altitude of 4 inches. Its volume is $\underline{\quad \frac{16\pi}{3} \quad}$ cubic inches.

28. A pyramid has an altitude which measures 4 inches, and a rhombus for a base whose side measures 4 inches and whose altitude measures $3\frac{1}{8}$ inches. Its volume is $\underline{\quad \frac{50}{3} \text{ or } 16\frac{2}{3} \quad}$ cubic inches.

Chapter 12

THE SPHERE

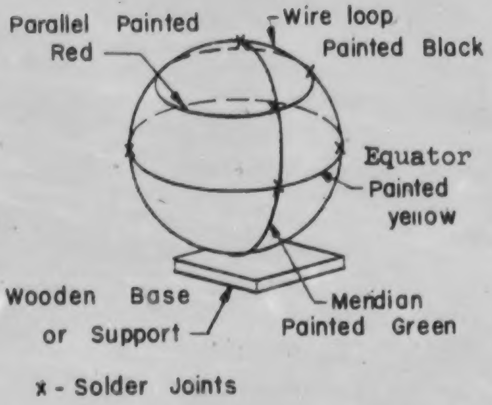
Ten or eleven days are recommended for this chapter.

The main purposes of this chapter are:

1. To develop enough of the properties of a sphere so that pupils will appreciate the significance of longtitude and latitude as a means of location of points on the surface of the earth.
2. To extend the use of proofs as a form of deductive reasoning. While many of the properties introduced in this chapter are not proved in a formal manner, two fundamental properties are proved in Section 3.
3. To introduce and use the formulas for the volume and area of a sphere. In this connection the teacher should find especially useful the chapter on the sphere in "Concepts of Informal Geometry" by R.D. Anderson (Studies in Mathematics, Volume 5 of the School Mathematics Study Group.)

Although many objects commonly used in everyday living are spherical we usually find that very little attention is given to the definition of a sphere or to the study of its properties. Indeed, the shape of the earth should be sufficient reason for devoting considerable attention to studying the sphere.

Many pupils display strong interest in the study of locating point on the surface of the earth. This interest can be motivated without introducing extremely difficult ideas about navigation. The teacher should capitalize on this interest as much as possible.



Teachers may find it helpful to encourage some pupils to make models of spheres. In Sections 3 and 4, it would be especially helpful to have a class size, wire model of the earth, using lengths of wire to represent the equator, parallels of latitude, meridians, etc. Perhaps a pupil who has great difficulty understanding properties of the sphere when these are presented in rather abstract fashion would find it especially helpful to construct such a wire model.

The number of days recommended for this chapter is 10 or 11. The teacher should perhaps be warned that the class should not be carried away by the geography to too great an extent. Of course if this material could be taken up in geography classes cooperatively, time could be saved and interest heightened in both courses.

12-1. Introduction

The purpose of this section is to start the pupils thinking in terms of a sphere. Each pupil should have as part of his equipment, a ball, or other spherical object, on which he can draw circles and over which he can place a string to trace paths of circles. Also, the classroom should be equipped with a large globe and a spherical solid. The spherical solid should be large enough for general classroom use and should be painted or covered in such a way that the teacher can draw lines on the surface. Such a sphere will be extremely useful in all sections of this chapter.

[pages 511-512]

Answers to Exercises 12-1

1. Baseball, basketball, softball, billiards, bowling, croquet, golf, lacrosse, jai-lai, soccer, tennis, marbles, polo, volleyball, water polo, ping pong, ball and jacks, handball, tetherball, medicine ball, beachball, cageball.
2. Storage tanks for petroleum products, balloons, as in toys, weather balloons, other planets, satellites.
3. A bobber, or float, may be spherical in shape when using a rod and line.
   Floats for nets are sometimes made of hollow, spherical metal containers. The U.S. Navy used huge spherical steel floats in hanging anti-submarine nets during war time.
4. Some light fixtures consist of a suspended sphere. Some valves used in pressure controls, etc., use ball bearings of spherical parts.
5. (a) circle
   (b) sphere
   (c) no thickness
6. For instance, a solid cube.
7. A sphere is a set of points in space such that all points in the set lie at the same distance from a particular point C, called the center of the sphere.
8. (a) The set of points is the interior of the sphere.
   (b) The set of points is the exterior of the sphere.

12-2. Great and Small Circles

This section is chiefly concerned with the intersections of lines and planes with a sphere. It will be necessary to recall some of the material presented in chapters covered previously which deal with geometric ideas. It is expected that most pupils

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will accept without any question that the intersection of a plane and a sphere is a circle.

The exercises here are exploratory. Not only should they be worked on outside of class, but they should serve as a basis for much of the class discussion at this time. It would be appropriate to spend two, or perhaps three class days on this section since thinking about the surface of a sphere will be new to most of the pupils. The time spent here in informal consideration should save time later on.

The ideas of meridian and parallel of latitude are touched on lightly here because of their connection with great and small circles. Their use as a coordinate system of reference in locating points is discussed in Section 4.

Answers to Exercises 12-2

1. (a) Yes. Every point on a sphere has an antipode.

   (b) No. Any point on a sphere has only one antipode.

2. (a) AB.

   (b) It is farther from A to B.

   (c) No. There is no point on a straight line through the interior more distant than B because the circle with center at A and radius AB touches the great circle through A and B only at the point B. There are points on the great circle more distant than B measured on the great circle if we go the "long way" on the great circle. But to any of these points there is always a great circle path which is shorter than the distance from A to B.

   (d) Yes. Since A and B are antipodal points they are points on a line segment containing C. Any great circle must be on a plane containing C. Such a plane contains $\overline{AC}$ and thus contains $\overline{AB}$.

(e) No. A, C, and D are three points in a plane but are not in a line. Only one plane can contain three such points, and that plane in the drawing contains the great circle shown.

3. (a) An infinite number. Hence, more than any finite number you can name.
   (b) An infinite number.
   (c) No. Antipodal points are endpoints of the diameters of a circle. Hence, the plane of any circle on the sphere through these points contains the center of the sphere and hence, is the plane of a great circle.

4. (a) No. One example would be small circles on parallel planes intersecting a sphere. There are others.
   (b) Yes. Consider two particular great circles. Each of these lies on a plane containing the center of the sphere. The intersection of these planes is a line since the intersection is not empty (it contains C). This line contains a diameter which is also a diameter of each great circle. This diameter contains a pair of antipodal points. These points lie on each great circle. Hence, the two great circles intersect.

5. (a) One. A meridian is a semi-circle (half a great circle) from pole to pole, and thus, is a curve on only a half of the surface of a sphere.
   (b) One.
   (c) No. Parallel planes do not intersect. Hence, circles on two parallel planes cannot intersect.

6. (a) D is in the interior of the sphere.

(b) No. E might be in the interior of the sphere, that is, just beyond $\overline{AC}$ on the line, but it might be in the exterior of the sphere. E might also be on the sphere, antipodal to A.

(c) A line passing through the center of the sphere intersects the sphere in two distinct points which are antipodal. One of these is A. If B is on $\overline{AC}$ and on the sphere it must be point A or the antipode of A.

(d) Yes, by definition of antipodal points.

(e) $\overline{AC}$ is called the radius. $\overline{BC}$ is also a radius.

7. (a) The interior of a sphere is the set of all points in space including the center itself, such that the distance of each point from the center of the sphere is less than the radius.

(b) The exterior of a sphere is the set of all points in space such that the distance of each point from the center of the sphere is greater than the radius.

(c) A sphere is the set of all points in space such that the distance of each point from the center of the sphere is the radius (or, is equal to the radius).

12-3. Properties of Great Circles

Mathematically, this section is the most important in the chapter. Up to this point, the background was in the process of being developed. After this section, the chief concern is in the application of these results.

The shortest path in a plane between two points is along a straight line, and the shortest path on a sphere between two points is along a great circle; hence, the importance of great circles.

No attempt is made to prove either of these basic facts. Although this property of a great circle may not be as obvious as that for a straight line, some experimentation should make it seem reasonable. Pupils should be encouraged to experiment with lengths of string and a globe until they become convinced of this.

In many other respects, great circles on a sphere behave quite differently from straight lines on a plane. Any two non-antipodal points determine a great circle, but each great circle intersects every other great circle in two points. These are the fundamental facts proved in this section.

With these results, the pupils should be able to go back over the previous set of exercises with greater understanding. They should, then, be able to pin down the reasons for some of the results which may have seemed doubtful. This should be done before going on to the applications. It might be advisable for the teacher to pause briefly at this point and review the previous exercises.

Answers to Exercises 12-3

1. (a) Using a globe with diameter 12 inches, the distance is about $6\frac{1}{2}$ inches.
   (b) About $9\frac{1}{2}$ inches on the above globe.
   (c) The great circle path is shorter.
2. (a) About $8\frac{1}{2}$ inches.
   (b) About 12 inches.
   (c) (b) is longer than (a).
3. The "best route" asked for here would be a matter of opinion. There is no shortest route because these two points are antipodes. A safest route might be to travel directly north from Singapore. This route offers more possible "crash-landing" places as much of this route is overland.

4. The shortest path between two points where one is due north of the other is a great circle path through the North Pole.

5. (a) The results in Problem 1 may be used to verify the fact that a great circle route is shorter than a route following a small circle.

   (b) When traveling between any two points on the equator.

6. Yes. Three points in space determine a plane. Any plane containing three points on a sphere intersects the sphere. Since the intersection set of the plane is not the empty set or a set of one point it must be a circle. If the plane passes through the center the points are on a great circle of the sphere. Otherwise they are on a small circle.

*7. White. It was a polar bear because the camp would have to be close to the North Pole. If the camp were at the North Pole, he would go north from any point to get back to camp. A second possibility might be that he had his camp two hours walk north of a small circle whose length was 12 miles, in which case he would shoot the bear at the point where he started his eastward trip. A third possibility would be that the small circle had length 6 miles so he would go around twice. In fact, the small circle could be of length $\frac{12}{n}$ miles for any positive integer n. Except for the bear, the camp might have been in the vicinity of the South Pole but there are no bears in the Antarctic.

12-4. Locating Points on the Surface of the Earth

This section is concerned with setting up a coordinate system on a sphere. In order to have a coordinate system on a surface, it is necessary that each pair of coordinates locate a unique point and it is desirable that each point have a unique pair of coordinates. The results of the previous sections show that the usual system of longitude and latitude has these two properties except

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for the North and South Poles and points on the 180th meridian.

Because the system seems so obvious, the use of meridians and parallels of latitude in locating points on the earth is generally treated very casually. It is all too frequently assumed that because a pupil can locate points by this system he understands the fundamental properties involved. Or, there is no concern about such understanding. This section attempts to bring out several important ideas:

1. That such a system uses a reference line for longitude and a reference line for latitude. Any meridian may serve as a zero meridian, or reference line, but the Greenwich meridian has been designated for this purpose. Similarly, any parallel of latitude may be used as a reference line, but the equator has been designated, and with obvious advantage, as the line of 0° latitude.

2. Longitude and latitude help us locate points on various hemispheres, but there is a subtle difference in the notation used for longitude and latitude. For longitude we measure east and west from the zero meridian (Greenwich Meridian) to the 180° meridian. For latitude, we measure north or south of the equator, but stop at the poles (which may be thought of as the 90° parallel--except that they are points, not circles).

Students interested in astronomy may be encouraged to learn more about the Tropic of Cancer, Arctic Circle, etc. It might be interesting for some students to prepare reports describing how these lines came to be designated, how they received their names, and so on. Also, students interested in geography can extend their study of antipodal points. There might be some cooperation, at this point, with the teachers of science or social studies.

[pages 525-530]

Answers to Exercises 12-4

1. (a) About 74° W, 41° N.
   (b) About 88° W, 42° N.
   (c) About 122° W, 38° N.
   (d) About 0°, 52° N.
   (e) About 2° E, 49° N.
   (f) About 38° E, 56° N.
   (g) About 42° W, 23° S.
   (h) About 144° E, 37° S.

2. 0°, 52° N. Greenwich is on the zero meridian.

3. 42°E., 0°. Any point on the equator has a latitude of zero.

4. (a) The dividing line between U.S. [illegible] Soviet Troops in Korea was fixed at the 38th parallel by the Yalta and Potsdam conferences. After the Korean War the dividing line was set at approximately this line.

   (b) Most of the states have at least partial boundaries along parallels of latitude. Among these are:
      (1) Northern boundary of Pennsylvania at 42nd parallel;
      (2) Northern boundary of South Dakota close to 46th parallel;
      (3) Nebraska - Kansas boundary at the 40th parallel;
      (4) California - Oregon at 42nd parallel.

   (c) In 1844 the United States claimed from Great Britain the whole of the Pacific Coast as far as Alaska, that is, to the 54° 40' parallel of latitude. The slogan of those in Oregon territory was "fifty-four forty or fight", but in the Oregon treaty of 1846 the boundary was fixed at the 49th parallel.

   (d) The Missouri Compromise provided that, except in Missouri, there should be in the Louisiana Purchase no slavery north of the $36\frac{1}{2}^\circ$ parallel.

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(e) The Mason and Dixon Line was originally the boundary between Pennsylvania and Maryland which was by charter supposed to have been the 40th parallel but was in fact a little below that. The Line was eventually considered to be the boundary which separated "slave states" from "free states."

(f) The complete name for Ecuador is "La Republica del Ecuador" which is Spanish for "The Republic of the Equator."

5. (a) Buenos Aires, Argentina

   (b) Wellington, New Zealand

6. (a) Pusan, Korea

   (b) Madrid, Spain

7. (a) Buenos Aires is located on approximately the antipode of Pusan, Korea.

   (b) Wellington and Madrid are located on points which may be considered antipodes.

8. Answers will vary. For most of the cities located in the 48 states between Canada and Mexico, antipodal points will be located in the Indian Ocean between Africa and Australia.

9. The northernmost tip of Alaska.

10. (a) Reno is at about the 120th meridian and Los Angeles at the 118th. Hence, the meridian of Reno is west of that of Los Angeles.

    (b) A little west of Quito, Ecuador.

    (c) The latitude of Portland, Oregon is 46°; that of Seattle is 47°; that of San Francisco is 38°. Hence, Portland is the closest.

    (d) The latitude of London is 51°, of Madrid is 41° and Casablanca is 33°. Hence, Madrid is closest.

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*11. No. A point on the sphere determines the intersection of a parallel of latitude and a meridian. Only one point is determined by each such intersection.

*12. Yes. The North Pole and the South Pole. Both the poles have a latitude of 90°, one being south the other north. However, both poles may be thought of as having an infinite number of locations with respect to the zero meridian. Also any point on the 180° meridian would be both 180° W and 180° E.

*13. To find the antipodal location of a point without using a map or globe, observe the following:

(1) The latitude will be the same except for direction. That is, for 45° N, the antipode will have latitude 45° S.

(2) For longitude, compute the distance half way around the earth from that point. That is, the antipode must be on a meridian 180° from the point. Subtract the longitude from 180. The difference is labeled with the direction opposite to the original longitude. For example: to find the antipode meridian for 40° W. 180 - 40 = 140. The antipodal meridian is 140° E.

(a) 100° E., 25° N.

(b) 80° W., 65° S.

(c) 0°, 52° N.

Note: The answer to 17 (c) is the location of Greenwich, England.

*14. The Tropic of Cancer is approximately at $23\frac{1}{2}^\circ$ North. This angle is the angle of maximum tilt of the earth's axis and the Tropic of Cancer marks the maximum distance north where the sun can appear directly overhead. The Artic Circle is approximately the southern border of the "land of the midnight sun," that is, where on June 21st the sun does not set. Actually due to various inaccuracies and discrepancies this phenomenon exists somewhat below the Arctic Circle.

*15. The International Date Line coincides with the 180th meridian for about half its length. Elsewhere it only approximates this.

*16. The location of the Antipodes is about 180° W and 52° S. When it is midnight at Greenwich, it is noon at the Antipodes because the islands are "twelve hours around the earth." When it is the middle of summer in Greenwich, it is midwinter at the Antipodes because of the tilt of the earth. No.

12-5. Volume and Area of a Spherical Solid

In the beginning, some attention is given to a very rough estimate of the volume of a sphere by comparing it with that of the inscribed and circumscribed cubes. The latter is quite simple but the teacher may not want to lay much stress on the former. The proof for the formula for the volume of a sphere is given in the reference mentioned at the beginning of the commentary for this chapter. The teacher may want to show this to some bright student.

No attempt is made to justify the formula for the surface area. The following justification might be given. The volume of a pyramid is one third of the altitude times the area of the base. Hence, if the sphere were thought of as composed of pyramids with vertices at the center and with bases little curved regions on the

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sphere, it would seem reasonable that the following would hold:

$$V = \frac{1}{3} r \cdot A$$

where r is the radius of the sphere (altitude of the pyramids) and A is the surface area. Then since $V = \frac{4}{3}\pi r^3$, the formula for the area follows, since $V = \frac{4}{3}\pi r^3 = \frac{1}{3} \times 4\pi r^2 \times r = \frac{1}{3}\pi A$.

Answers to Exercises 12-5

1. (a) $113\frac{1}{7}$ cu. in.

   (b) $4190\frac{10}{21}$ cu. ft.

   (c) $268\frac{4}{21}$ cu. yd.

   (d) $905\frac{1}{7}$ cu. cm.

   (e) 735.91 cu. in.

   (f) 1204.75 cu. in.

   (g) 2483.71 cu. mm.

   (h) 310.46 cu. ft.

   Answers to (e) - (h) are given correct to two decimal places.

2. Answers to the following are given correct to no more than two decimal places.

   (a) 113.04 sq. in.

   (b) 1256 sq. ft.

   (c) 200.96 sq. yd.

   (d) 452.16 sq. cm.

   (e) 393.88 sq. in.

   (f) 547.11 sq. in.

   (g) 886.23 sq. mm.

   (h) 221.56 sq. ft.

3. (a) $r = 25$, surface area = 7850 sq. ft., number of gallons of paint is 19.625, cost \$ 117.75.

   (b) Volume is approximately 65,400 cu. ft., number of gallons of oil is approximately 491,000 and the value of the oil is approximately \$ 63,800.

4. Volume of the bowl is 1024 cu. in., the weight of the sugar is 32 pounds and mother pays \$ 3.20.

5. r is 9 in., the surface area of one globe is 1017.36 sq. in., and the total cost of the plastic is $353.25.

6. $r = 20$ ft., the volume of the gas is 33493.33 cu. ft.

7. (a) The volume is multiplied by 8 since

$$\frac{4}{3}\pi(2r)^3 = \frac{4}{3}\pi(8r^3) = 8\left(\frac{4}{3}\pi r^3\right).$$

The surface area is multiplied by 4, since

$$4\pi(2r)^2 = 4\pi(4r^2) = 4(4\pi r^2).$$

(b) The volume is multiplied by 27 and the surface area by 9, for similar reasons.

8. Call the radii R and r. Then $\frac{R}{r} = \frac{3}{2}$.

(a) The volumes are $\frac{4}{3}\pi R^3$ and $\frac{4}{3}\pi r^3$ and so

$$\frac{\text{Volume of one sphere}}{\text{Volume of other sphere}} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi r^3} = \left(\frac{R}{r}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

(b) The surface areas are $4\pi R^2$ and $4\pi r^2$, so

$$\frac{\text{Surface area of one}}{\text{Surface area of the other}} = \frac{4\pi R^2}{4\pi r^2} = \left(\frac{R}{r}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

12-6. Finding Lengths of Small Circles

This section is not an essential one, but it gives a pupil the means of computing the lengths of circles of latitude given in Section 12-4. It also gives him a chance to use again the trigonometry which was developed in Chapter 9.

Answers to Exercises 12-6

1. (a) 24150 miles
   (b) 6470 miles
   (c) 17680 miles

2. (a) 21650 miles
   (b) 10825 miles
   (c) 10 degrees
   (d) $\frac{10}{180}$
   (e) $\frac{10}{180}$ (10825 miles) ≈ 600 miles
   (f) $\frac{600}{400}$ hrs. = 1.5 hrs. = 1 hr. 30 min.

3. (a) Distance ≈ 3190 miles.
   (b) Distance ≈ 4260 miles.

*4. The earth makes one-half of a revolution in 12 hours. We might say that the longitude of the point on the earth "directly under" the sun changes by 180° in 12 hours. Hence, the longitude change is 15° per hour, and the answers are:

   (a) Since the difference in longitude is 60° the sun-time difference is 4 hours. Thus the sun-time at 70° W is 3:00 a.m.

   (b) Since the difference in longitude is 80° the sun-time difference is $\frac{80}{15}$ hours, or 5 hours and 20 minutes. Thus the sun-time at 10° E is 12:20 p.m.

*5. The length of the circle of latitude at 40° is approximately $25000(\cos 40^\circ) \approx 19150$ miles.
This will be the approximate distance the sun moves relative to the earth at this latitude in 24 hours. Hence, the distance for one hour's difference in sun-time will be

$$\frac{19150}{24} \approx 800 \text{ miles.}$$

A is approximately 800 miles west of B.

6. The following results were obtained by using a regulation softball and standard half-gallon, paper milk carton.

Carton:	Sphere:
$\ell \approx 3\frac{3}{4}''$	$d \approx 3\frac{10}{16}''$
$w \approx 3\frac{3}{4}''$	$r \approx 1\frac{13}{16}''$ or $\frac{29}{16}''$
Water level raised $\approx 1\frac{3}{4}''$	
Volume ≈ 24.6 cu. in.	$r^3 \approx \frac{24389}{4096} \approx 5.9$

Then,

$$\pi r^3 \approx \frac{536558}{28672} \approx 18.0$$

(using $\frac{22}{7}$ for π)

$$18.0 < 24.6$$

$24.6 \div 18.0 \approx 1.36$, showing that the constant is about $\frac{4}{3}$.

Sample Questions for Chapter 12

True-False

T 1. All points on a parallel of latitude are the same distance from the North Pole.

F 2. The North Temperate Zone on the earth's surface is bounded by two great circles.

F 3. All points on the zero meridian are the same distance from the equator.

T 4. All diameters for a given sphere have the same length.

T 5. For a given sphere, the center of a great circle of the sphere is the center of the sphere.

F 6. All small circles of a given sphere have equal lengths.

F 7. For a given sphere, two great circles cannot have the same center.

T 8. The diameter of any meridian is an axis of the sphere.

T 9. All great circles passing through the North Pole must pass through the South Pole.

F 10. The intersection set of a line and a sphere must contain two points, assuming the set is not the empty set.

T 11. A plane which intersects a sphere in only one point is said to be tangent to the sphere.

T 12. The radius of a given sphere is R. If a point X is located a distance G from the center of the sphere, and $G > R$, then X is in the exterior of the sphere.

T 13. If the radii of two circles of a given sphere are not equal, then not both the circles are great circles of the sphere.

F 14. Parallels of latitude lie on planes which are parallel to the axis of the earth.

F 15. The longitude of the North and South Pole is usually written as 90°.

T 16. The International Date Line follows part of the 180° meridian.

T 17. The cities of New York and London are located as points on a globe. Only one great circle may pass through these two points.

F 18. If a point is located at X 100° E., 0°, its antipode is located at 100° W., 0°.

T 19. Every great circle of a sphere has its center at the center of the sphere.

F 20. The line through two centers of small circles of a sphere must pass through the center of the sphere.

F 21. Through any two points of a sphere there is just one small circle.

F 22. Through any two points of a sphere there is at least one small circle.

F 23. If you double the radius of a sphere, you double its surface area.

T 24. If you double the radius of a sphere you multiply its volume by 8.

T 25. Suppose the center of one small circle is just as far from the center of the sphere as the center of another small circle. The lengths of the circles must be equal.

Multiple Choice

B 1. Read the following statements A, B, C, D.

I. The area of a sphere is equal to four times the area of a great circle of the sphere.

II. The area of a sphere is equal to eight times the area of any small circle of the sphere.

III. The formula for the area of a sphere is $A = 4\pi r^2$.

IV. The formula for the area of a sphere is $A = \pi r^2 h$.

Which of the following (1, 2, 3, 4, 5) is true?

A. Only I is true.

B. Only I and III are true.

C. Only II and III are true.

D. Only II and IV are true.

E. All of the statements are true.

D 2. The formula for the volume of a sphere is:

A. $V = 4\pi r^2$

B. $V = \pi r^2 h$

C. $V = \pi r^3$

D. $V = \frac{4}{3} \pi r^3$

E. $v = 2\pi r^2 h$

E 3. Consider the following four statements:

I. All radii are congruent.

II. All diameters are congruent.

III. The length of a diameter is equal to twice the length of a radius.

IV. All great circles have the same length.

Which of the following are true for one specific sphere?

A. Only I and II are true.

B. Only III and IV are true.

C. Only I and III are true.

D. Only I, II, and III are true.

E. All of the statements are true.

C 4. A sphere has a radius r. Point A is located a distance x from the center of the sphere. If A is in the exterior of the sphere, which one of the following must be true?

A. $x = r$

B. $x < r$

C. $x > r$

D. $x = 2r$

E. None of the above answers is correct.

D 5. If the radius of a sphere is doubled, the volume of the original sphere is multiplied by:

A. two

B. four

C. six

D. eight

E. ten

E 6. The volume of a sphere whose radius is 3 units is ...

A. 3π

B. $(3\pi)^2$

C. 9π

D. 12π

E. 36π

C 7. What is the length of a great circle of a sphere whose diameter is 10 inches?

A. About 20 inches.

B. About 25 inches.

C. About 30 inches.

D. About 35 inches.

E. About 40 inches.

B 8. On the earth, the distance from the North Pole to the point located at 76° W., 0° is about ...

A. 4000 miles

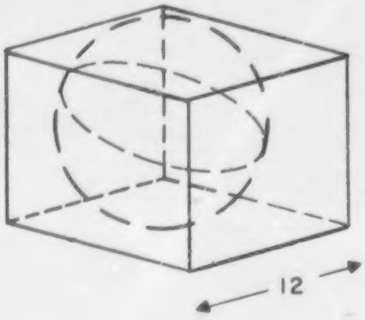
B. 6000 miles

C. 10,000 miles

D. 12,000 miles

E. 25,000 miles

C 9. In the figure at the right the sphere is inscribed in a cube whose edge has a length of 12 inches. The volume of the sphere is nearest:



A. 300 cu. in.

B. 500 cu. in.

C. 1,000 cu. in.

D. 1,500 cu. in.

E. 2,000 cu. in.

A 10. The surface of the sphere shown in the drawing for Question 9 is about:

A. 450 square inches

B. 550 square inches

C. 600 square inches

D. 650 square inches

E. 700 square inches

C 11. The radius of sphere A is twice as long as the radius of sphere B. The surface area of sphere A is how many times that of sphere B?

A. 2

B. 3

C. 4

D. 8

E 10

D 12. A and B are on two points on earth such that B is directly north of A. If A and B move due west, what can be said of their paths?

A. Their paths will cross.

B. Their paths will be on the same line.

C. Their paths will move in opposite directions.

D. Their paths will not cross.

E. Their paths would cross if continued around the sphere twice.

B 13. A is on a point located at 100° W., 25° N. B is on a point located at 120° W., 25° N. If A and B move due south, then:

A. Their paths cannot intersect.

B. Their paths will intersect.

C. Their paths will move them farther apart.

D. Their paths will intersect in a line.

E. Their paths will intersect in many separate points.

B 14. Which one of the following cities has the greatest measure for latitude?

A. New York City

B. London

C. Madrid, Spain

D. Tokyo

E. San Francisco

D 15. A formula for the total surface of a hemisphere (that is, the curved surface and the flat region) is ...

A. πr^2

B. $2\pi r^2$

C. $2\frac{1}{2}\pi r^2$

D. $3\pi r^2$

E. $\frac{4}{3}\pi r^2$

A 16. The radius of the earth is about 4,000 miles and the radius of the moon is about 1,000 miles. Which of the following ratios describes how the area of the earth compares with the area of the moon?

A. 16:1

B. 12:1

C. 8:1

D. 4:1

E. 2:1

Chapter 13

WHAT NOBODY KNOWS ABOUT MATHEMATICS

13-1. Introduction

The object of this chapter is inspiration, not perspiration. Its idea is to make clear that mathematics is a living, growing, active subject and to stimulate the pupils with a glimpse of some of the open questions. The problems are exploratory, not designed for developing specific skills, and should not be made unpleasant drudgery. It is hoped that some of the questions will excite curiosity. It must be kept in mind, however, that no real insights are obtained without some effort. As much as possible encourage the students to explore and make conjectures on their own. Some of the topics here could easily be related to projects for science fairs and junior academics of science.

The topics in the chapter have been arranged to try to give variety in types of applications. No particular sequence order is involved and it would be perfectly feasible to take some sections and omit others altogether.

The last section seeks to inject the thought of mathematics as a career. It might be worthwhile to cap this with a speech to the class by a mathematician from a nearby college, industry, or government agency.

Supplementary reading related to this chapter or about various mathematicians may be found in Bell, Men of Mathematics, and Boehm, The New World of Mathematics. See also the references mentioned in the text.

13-2. A Conjecture about Primes

This section reminds the pupil of known properties of primes and tries to guide him by discovery methods to the Goldbach Conjecture. This is a good example of the formulation of a conjecture on the basis of observation and experience, using the inductive method (see Volume I, Chapter 1), as is frequently the case in mathematics. It also furnishes an illustration of the fact that experimentation must not be taken as proof.

The first attempt to write numbers from 4 to 20 as a sum of two primes would lead to

$4 = 2 + 2$
$5 = 2 + 3$
$6 = 3 + 3$
$7 = 2 + 5$
$8 = 3 + 5$
$9 = 2 + 7$

$10 = 3 + 7$
$ = 5 + 5$
$11 = \underline{\qquad\quad}$
$12 = 5 + 7$
$13 = 2 + 11$
$14 = 7 + 7$
$ = 3 + 11$

$15 = 2 + 13$
$16 = 3 + 13$
$ = 5 + 11$
$17 = \underline{\qquad\quad}$
$18 = 5 + 13$
$ = 7 + 11$
$19 = 2 + 17$
$20 = 3 + 17$
$ = 7 + 13$

where 11 and 17 cannot be written as a sum of two primes.

The part of the table which is not already included above is:

$22 = 3 + 19 = 5 + 17 = 11 + 11$
$24 = 5 + 19 = 7 + 17 = 11 + 13$
$26 = 3 + 23 = 7 + 19 = 13 + 13$

Answers to Exercises 13-2a

Problems 1 and 2. The table on the following page shows in the first column the even numbers from 4 to 100, in the third column the number of representations as the sum of two primes, and in the second column for each such representation the smaller of the two primes. Thus, since $10 = 3 + 7 = 5 + 5$ we list in the second column the primes 3, 5 and in the third the number of representations 2. The answer to both Questions 1 and 2 can be read at once from the table.

Even number	The smaller of the two primes in each representation	Number of pairs
4	2	1
6	3	1
8	3	1
10	3, 5	2
12	5	1
14	3, 7	2
16	3, 5	2
18	5, 7	2
20	3, 7	2
22	3, 5, 11	3
24	5, 7, 11	3
26	3, 7, 13	3
28	5, 11	2
30	7, 11, 13	3
32	3, 13	2
34	3, 5, 11, 17	4
36	5, 7, 13, 17	4
38	7, 19	2
40	3, 11, 17	3
42	5, 11, 13, 19	4
44	3, 7, 13	3
46	3, 5, 17, 23	4
48	5, 7, 11, 17, 19	5
50	3, 7, 13, 19	4
52	5, 11, 23	3
54	7, 11, 13, 17, 23	5
56	3, 13, 19	3
58	5, 11, 17, 29	4
60	7, 13, 17, 19, 23, 29	6
62	3, 19, 31	3
64	3, 5, 11, 17, 23	5
66	5, 7, 13, 19, 23, 29	6
68	7, 31	2
70	3, 11, 17, 23, 29	5
72	5, 11, 13, 19, 29, 31	6
74	3, 7, 13, 31, 37	5
76	3, 5, 17, 23, 29	5
78	5, 7, 11, 17, 19, 31, 37	7
80	7, 13, 19, 37	4
82	3, 11, 23, 29, 41	5
84	5, 11, 13, 17, 23, 31, 37, 41	8
86	3, 7, 13, 19, 43	5
88	5, 17, 29, 41	4
90	7, 11, 17, 19, 23, 29, 31, 37, 43	9
92	3, 13, 19, 31	4
94	5, 11, 23, 41, 47	5
96	7, 13, 17, 23, 29, 37, 43	7
98	19, 31, 37	3
100	3, 11, 17, 29, 41, 47	6

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Generally speaking the number of representations as a sum of two primes goes up with the size of the number, but there are many ups and downs in the third column. In particular, a few numbers like 38, 68, and 98, have surprisingly few representations.

13-3. Distribution of Primes

This section tries to point out some interesting properties about the sequence of primes, especially those related to the gaps in the sequence.

Since the difference of any two odd numbers is always even, the difference of any two consecutive primes greater than 2 must be even. The first occurrences of gaps of 2, 4, 6, 8, 10, 12, and 14, respectively, in the sequence of primes are:

$$\begin{aligned} 2 &= 5 - 3 \\ 4 &= 11 - 7 \\ 6 &= 29 - 23 \\ 8 &= 97 - 89 \\ 10 &= 149 - 139 \\ 12 &= 211 - 199 \\ 14 &= 127 - 113 \end{aligned}$$

These results can be obtained by examining a list of primes.

If the discussion of Euclid's proof about gaps in the sequence of primes seems hard for the pupils it may be well to consider first some simpler cases where the results can actually be verified. For example, you may point out that the numbers $(2 \cdot 3 \cdot 4) + 2$, $(2 \cdot 3 \cdot 4) + 3$, $(2 \cdot 3 \cdot 4) + 4$, are three consecutive non-primes and $(2 \cdot 3 \cdot 4 \cdot 5) + 2$, $(2 \cdot 3 \cdot 4 \cdot 5) + 3$, $(2 \cdot 3 \cdot 4 \cdot 5) + 4$, $(2 \cdot 3 \cdot 4 \cdot 5) + 5$, are four consecutive non-primes. Applying Euclid's reasoning to one of these cases first may make matters more clear.

You may note that Euclid's proof in the last case above shows that numbers 122, 123, 124, 125 are consecutive non-primes, so there is a gap of at least four in the sequence of primes at this point. Actually the gap in the sequence of primes which contains these numbers is the one from 113 to 127, a gap of 14. Thus, Euclid's method may be a very poor estimate of the actual size of the gap.

Somewhere in this work on properties of numbers you may find it interesting to tell this story of the Indian mathematician, Ramanujan, of whom it was said that every number was a personal friend. Ramanujan was extremely ill and his friend, the English mathematician, Hardy, was visiting him. Hardy tried to cheer him up with small talk and remarked, "On my way here I saw a car with a very uninteresting license number - 1729." Ramanujan replied like a flash, "On the contrary that is a most interesting number. It is the smallest number which is representable in two ways as a sum of two cubes

$$1729 = 1^3 + 12^3 = 9^3 + 10^3."$$

Answers to Exercises 13-3a

1 and 2. For Problem 1 it will be necessary to refer to the table of primes up to 1000. It may be well to do these problems as a class project. This work will be tedious and time consuming for any one individual, especially the calculations asked for in Problem 2. The table on the following page gives the data asked for in Problem 1 and also shows in the last column the figures asked for in Problem 2.

n	Number of primes < n	Number of twins < n	$\frac{\text{Number of twins} < n}{\text{Number of primes} < n}$	Product of 1st and 3rd columns divided by the square of the second
100	25	15	$\frac{15}{25} = .60$	$\frac{(100)(15)}{25^2} = 2.40$
200	46	29	$\frac{29}{46} \approx .63$	$\frac{(200)(29)}{46^2} \approx 2.57$
300	62	37	$\frac{37}{62} \approx .60$	$\frac{(300)(37)}{62^2} \approx 2.9$
400	78	41	$\frac{41}{78} \approx .54$	$\frac{(400)(41)}{78^2} \approx 2.7$
500	95	47	$\frac{47}{95} \approx .50$	$\frac{(500)(47)}{95^2} \approx 2.6$
600	109	52	$\frac{52}{109} \approx .47$	$\frac{(600)(52)}{109^2} \approx 2.6$
700	125	59	$\frac{59}{125} \approx .47$	$\frac{(700)(59)}{(125)^2} \approx 2.65$
800	139	59	$\frac{59}{139} \approx .42$	$\frac{(800)(59)}{(139)^2} \approx 2.45$
900	154	69	$\frac{69}{154} \approx .45$	$\frac{(900)(69)}{(154)^2} \approx 2.6$
1000	168	69	$\frac{69}{168} \approx .41$	$\frac{(1000)(69)}{(168)^2} \approx 2.45$

In Problem 1, column 4 makes it appear likely that the ratio of the number of twin primes to the total number of primes less than some number n, decreases as n gets larger. From the table we see that the ratio has decreased from 0.60 to 0.41 as n increased from 100 to 1000. It was proved by a Norwegian mathematician, Viggo Brun, that this decrease in the ratio continues until it is less than 0.10, less than 0.01, less than 0.001, etc. until it actually approaches zero. Of course, it can never reach zero since the ratio always has a number other than zero as its numerator. We express this result by saying that zero is the limit of the ratio of the number of twin primes less than a number n to the total number of primes less than n, as the number n increases without bound.

For Problem 2, an inspection of the last column on the previous page might lead one to conjecture that these numbers approach some definite limit as n grows larger and larger. This has not been proved, however.

3. Below is the required table. As a matter of interest it has been extended to about twice the number of entries actually asked for in the problem.

Interval	Number of Primes	Interval	Number of Primes
1-4	2	256-289	7
4-9	2	289-324	5
9-16	2	324-361	6
16-25	3	361-400	6
25-36	2	400-441	7
36-49	4	441-484	7
49-64	3	484-529	7
64-81	4	529-576	6
81-100	3	576-625	9
100-121	5	625-676	8
121-144	4	676-729	7
144-169	5	729-784	8
169-196	5	784-841	9
196-225	4	841-900	8
225-256	6	900-961	8

The figures suggest that the number of primes between successive perfect squares tends to increase as the squares get larger. Whether this is true is not known. As noted in the student book, it is not even known whether there is always one prime between any two perfect squares.

4. Here is the required table.

n	Number of primes of form $4k + 1$ which are less than n	Number of primes of form $4k + 3$ which are less than n
10	1	2
20	3	4
30	4	5
40	5	6
50	6	8
60	7	9
70	8	10
80	9	12
90	10	13
100	11	13
110	13	15
120	14	15
130	14	16
140	15	18
150	16	18
160	17	19
170	17	21
180	18	22
190	19	22
200	21	24

The table suggests the conjecture that for any n there are more primes of the form $4k + 3$ than primes of form $4k + 1$. Notice that this does not prove the conjecture.

5. The three consecutive odd numbers which are primes are, of course, 3, 5, 7. No other example appears in the list of primes as far as you have written them. The proof that no other set of three consecutive odd numbers can all be prime goes like this. Consider any three consecutive odd numbers. Any odd number is one more than an even number, so can be written $2k + 1$. The next two odd numbers are found by adding 2 and 4 to this one, so the three consecutive odd numbers can be written as

$$2k + 1, \quad 2k + 3, \quad 2k + 5.$$

Now when any number is divided by 3, the remainder must be either 0 or 1 or 2. In particular, if $2k$ is divided by 3, it must have one of these three remainders. Let us look at each of the three possibilities.

(a) Suppose the remainder is 0. Then $2k = 3r$ for some r. In this case, $2k + 3 = 3r + 3 = 3(r + 1)$, where we have used the distributive law. This shows $2k + 3$ has a factor of 3, so the middle one of the three numbers is divisible by 3.

(b) Suppose the remainder is 1, so that $2k = 3r + 1$. Then $2k + 5 = (3r + 1) + 5 = 3r + 6 = 3(r + 2)$, so in this case, the third number is divisible by 3.

(c) Suppose the remainder is 2, so that $2k = 3r + 2$. Then $2k + 1 = (3r + 2) + 1 = 3r + 3 = 3(r + 1)$, so in this case, the first number is divisible by 3.

Thus, one of the three consecutive odd numbers must be divisible by 3. But a number divisible by 3 cannot be prime unless it is the number 3 itself. Thus, the only possibility of having 3 consecutive odd prime numbers is for one of the numbers to be 3. This occurs only in the case of 3, 5, 7 which we already knew, so we have shown there are no other cases.

This problem probably will be hard even for the best students and should be used as a special challenge problem, not as one for the entire class to try.

13-4. Problems on Spheres

The material here should be reasonably clear from the student text book. It is hoped that actual handling of both the disks and spheres will develop real geometric insight into the meaning of these packing problems. Some experimental work with packing before reading the text would be good. If a reference is desired, the problem is discussed briefly on pp. 148-151 of Rouse Ball, Mathematical Recreations and Essays (revised by Coxeter).

One point should be emphasized. In an actual packing either of disks in a region of the plane or spheres in a region of space, there is almost always waste space around the edges of the region. For example, in Problem 1 below, where disks are being packed into a rectangular region, every other row will have about half a disk of waste space at one end or the other. This is the reason why the actual results on the part of the area covered by disks is less than the theoretically computed result. As the region gets larger and larger, this boundary waste is a smaller and smaller fraction of the whole region and the actual results get closer and closer to the theoretically computed ones.

Answers to Exercises 13-4

1. For the 8×8 square, you seem to be able to get in 68 pennies and the ratio of area of circles to area of region is about 0.834.
   For the 14×14 square, you seem to get in 216 pennies and a ratio of 0.865.

2. Each disk has an area of $\frac{\pi}{4} \approx .785$ square units. Since at most 90.7% of the area can be covered by the disks, the total area of the disks is at most $(0.907)(1000) = 907$ square units. But $\frac{907}{0.785} = 1155.4$. Thus, the number of disks, which is necessarily a whole number, cannot exceed 1155.

3. The answer will depend on the size of the box and the care with which packing is done. In practice the ratio will probably be well under 0.74.

4. Twelve. Each interior sphere in the packing touches six spheres in its own layer and touches three spheres of the layer above and three of the layers below. This is clear from the fact that in placing a second layer of spheres, there are three of the "pockets" around any sphere of the first layer in which we place spheres of the second layer.

If your pupils seem particularly interested in such geometric problems, you may wish to mention the following problem which is unsolved, but not too hard to understand.

How should n points be placed on the surface of a sphere of radius 1, so that the smallest of the distances between pairs of points is as large as possible? You can visualize this problem by imagining that you have n ants on the surface of the sphere who hate each other enthusiastically and want to stay as far apart as possible. The question is how they would arrange themselves on the sphere. For a few of the simpler cases the solution is easy to see. For example, when $n = 2$ the two points would be placed at opposite points of a diameter. The problem was solved in 1951, by Schmitte and Van der Waerden for the cases, $n = 5, 6, 7, 8, 9, 12$. But even for $n = 10$ the solution is not known.

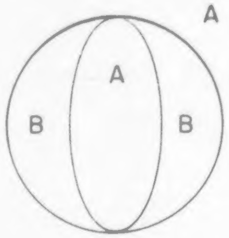
13-5. A Problem on Coloring Maps

The intention here, actually is to introduce the problem and make clear what it is, rather than to go into detail on all that is known about it. However, it is a very appealing problem to some people. For anyone who wants to read a little more about map colorings, the two following expository articles may be worth looking up.

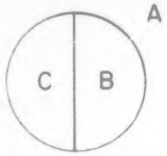
P. Franklin, *The Four-Color Problem*, Galois Lectures, Scripta Mathematica Library,# 5.

H. R. Brahana, "The Four-Color Problem," *American Mathematical Monthly*, 30, 1923, p. 234.

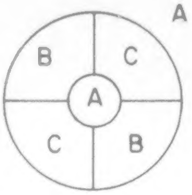
For the map shown here the coloring can be done in 2 colors, as shown, where the letters A, B denote any two distinct colors, because any two regions having a boundary arc are of opposite colors.

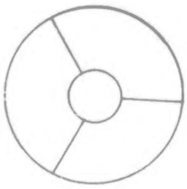


The map here requires 3 colors, as shown, for each two of the three regions have an arc in common, so no two can be the same color.



The coloring of may be done in three colors, as shown.





But the map

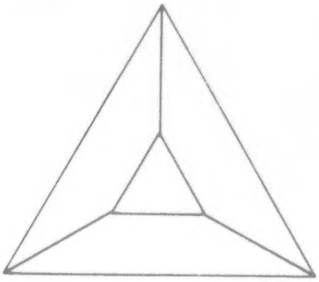
cannot be colored in less than four colors, since, as noted in the student text, the inner circle and the three countries of the ring all have arcs in common, so no two can be colored the same. Hence, at least four colors are necessary.

The suggested attempt to draw a map with five countries, each of which has a boundary arc in common with the other four, proves nothing and was not intended to. However, some experimentation will fairly quickly convince the pupil that there is a fundamental difficulty. It appears that whenever you draw four countries, such that each two have a boundary arc in common, somehow three of them seem to isolate the fourth so that you cannot draw a fifth country touching this one as well as touching the first three. This fact has, of course, been guessed, not proved, but does at least serve to motivate the question about actually needing five colors for some maps.

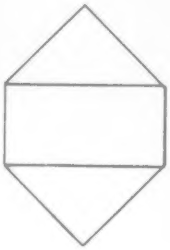
Answers to Exercises 13-5a

1. There are infinitely many ways of distorting the original map into a polygonal map. The following are possible answers.

(a)

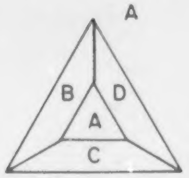


(b)



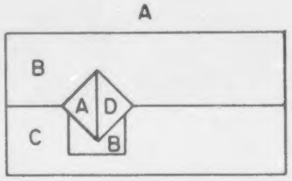
2. Shown below are color schemes for each map using the smallest number of colors. The notation means that all countries marked A are to be colored one color, all those marked B a different color, etc.

(a)



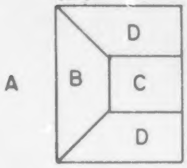
4 colors

(b)



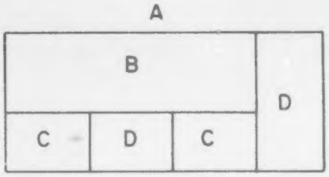
4 colors

(c)



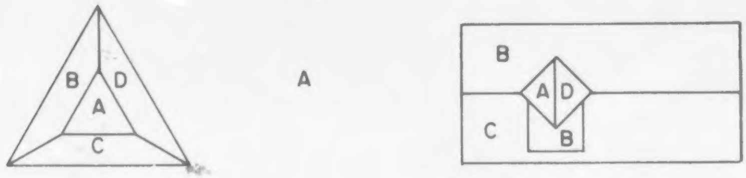
4 colors

(d)



4 colors

3.



Since the scheme needs only 4 letters, the coloring uses only 4 colors.

4. If you know how to color each map separately, you have a color scheme for each, as we had in the special case above. Of course, in any coloring scheme you may interchange two letters and get another color scheme. If necessary make a change of letters in one map, so that the "ocean" is marked with the same letter. Then simply use the combined color scheme as above. What we are really saying, is that you begin by deciding on a color for the ocean, then go ahead and color each island as though the other weren't there at all.

The next section on the number of ways of coloring a map is a good exercise in permutations, reviewing the ideas in Chapter 7, but is no essential part of the development, and may be omitted at will.

Answers to Exercises 13-5a (continued)

5. (a) The regions marked A, B, C, D each have arcs in common with each of the others, so all these must be different colors, as shown. Then X can be chosen as any color except those of the adjoining regions, here the regions marked B, C, D. Similarly, Y can have any color except those colors of the adjacent regions, A, B.

The argument is essentially the same for Parts (b) and (c).

6.

n	Map 2(a)	Map 2(b)	Map 2(c)	Map 2(d)
3	0	0	0	0
4	24	48	24	24
5	240	720	240	480
6	1080	4320	1080	3240

7. The required number of ways for any number n of colors was shown to be $n(n-1)(n-2)(n-3)^2$. For $n = 3, 4, 5, 6$ this gives the column in Problem 6 for Map 2(a).

8. Map 2(b) $n(n-1)(n-2)^2(n-3)^2$

Map 2(c) $n(n-1)(n-2)(n-3)^2$

Map 2(d) $n(n-1)(n-2)(n-3)^3$

Notice that the expressions above are the same for Maps 2(a) and 2(c); this shows why these two columns are the same in Problem 6.

The material in and before Exercises 13-5b is designed to lead to an empirical discovery of Euler's formula on maps without islands.

Answers to Exercises 13-5b

1.

	Map 2(a)	Map 2(b)	Map 2(c)	Map 2(d)
V	6	14	8	12
C	5	6	5	6
E	9	18	11	16

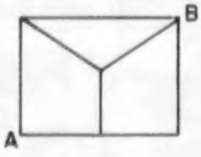
2, 3, 4, 5. The purpose is for the student to discover the relation $V + C = E + 2$ or equivalently,

$$V + C - E = 2.$$

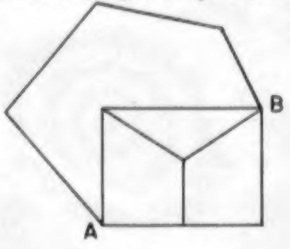
Before the pupil discovers this he may make some other simpler observations, such as that $V + C + E$ is even. In this case, you should commend his observation, but tell him there is more to find. If there is difficulty in discovering the relation, suggest that in the tables the pupils add a fourth row showing $V + C$.

A formal proof of Euler's formula, probably is not possible or desirable here. However, the following reasoning may give a little insight for the pupils who want to know why the relation is true.

Suppose we consider a polygonal map and make a new map by drawing an additional path joining two vertices. For example, Map 1 below, is turned into Map 2 by drawing the path shown from A to B.



Map 1



Map 2

[pages 570-571]

Let us compare the numbers of vertices, countries, and edges, for the two maps. Map 2 has exactly 4 more edges than Map 1, since the path was made up of 4 segments. It has 3 more vertices, since a 4 segment path must have three vertices in addition to the end-points. Finally, Map 2 has just one more country, since the path A, B divided what had been one country into two countries. If we make a table of the vertices, edges, and countries for these maps it looks like this:

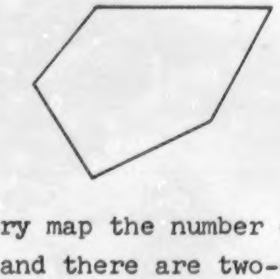
	Map 1	Map 2
V	6	6 + 3 = 9
C	4	4 + 1 = 5
E	8	8 + 4 = 12.

But then for Map 2,

$$V + C - E = (6 + 3) + (4 + 1) - (8 + 4)$$

$$V + C - E = (6 + 4 - 8) + (3 + 1 - 4) = 6 + 4 - 8$$

That is, V + C - E has the same value for Map 2, that it does for Map 1, since the number of edges added (4) exactly equals the sum of the number of countries added (1) and the number of vertices added (3). This illustrates that if we make step by step additions to a map by drawing additional paths, the new maps all have the same value for V + C - E. Now, any map can be obtained by starting with a two-country map, such as that below and making step-by-step additions as above.



But for a two-country map the number of edges is equal to the number of vertices and there are two-countries, so V + C - E = 2. Thus, it seems clear that for all maps (without islands)

V + C - E = 2. The basic idea of this reasoning can actually be made into a real proof, but we will satisfy ourselves with this.

Remember that all this work has been for maps without islands or perhaps we should say that so far each map *is* one island in an ocean. Some of your students may be interested to look at the case when several islands are allowed in the map, that is, the map falls apart into separate pieces as suggested in Problems 3, 4 of Exercises 13-5a. It turns out that if there are I islands the relation becomes

$$V + C - E = 1 + I.$$

Interesting extensions of these questions concern maps drawn on other surfaces, such as the sphere, the torus (doughnut), etc.

13-6. The Travelling Salesman

The reference to the solution of the travelling salesman problem is G. Dantzig, R. Fulkerson, S. Johnson, *Solution of a Large-Scale Travelling Salesman Problem*, Journal of Operations Research Society of America, Vol. II, pp. 393-410, (1954).

A general discussion is given on pp. 340-357 in J.F. McCloskey and J.M. Clippinger, *Operations Research for Management*, Vol. II. This book gives other applications of mathematics to business and related problems.

Another interesting book, dealing with somewhat similar problems is, as mentioned in the student text, J.D. Williams, *The Compleat Strategyst*.

[pages 570-573]

Answers to Exercises 13-6

1. ABCDA or ADCBA = 930 miles.
2. ABDCA and ADCBA both give an answer of 10.
3. 3,628,800
4. (a) 6 ways (b) $5 + 8 + 6 = 19$
5. (a) 24 ways

   (b) $$\left.\begin{array}{l}7 + 13 + 6 + 16\\ 7 + 13 + 8 + 14\\ 5 + 15 + 6 + 16\\ 5 + 15 + 8 + 14\end{array}\right\} \text{all equal } 42$$

13-7. Applied Physical Problems

The essential purpose here is to suggest to the pupils the large number of physical situations which lead to interesting mathematical problems. This is one of the most fruitful sources of mathematical problems, although the pupils do not yet have the background to see how the physical situations are placed in mathematical form. Hence, this section has been kept both brief and wholly descriptive.

13-8. Conclusion

The main purpose of this section is to emphasize that mathematics not only is being done, but is being done by human beings and not handed down from some remote source on Mt. Olympus. Pupils are often given no inkling that they themselves might sometime do mathematical work. One of the more horrifying aspects of our culture is the high school student's mental picture of the scientist as reported by a recent survey conducted by *Fortune Magazine*. Occasionally, some human interest stories about mathematicians, such as, Euler's blindness or Descartes lying in bed till noon, may help dispel the Olympian fixation. Basically, however, we hope this chapter will have given a vista of an active and fascinating field.

[pages 573-579]